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On Leonardo Pisano dual quaternions

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Abstract

In this work, firstly we introduce the Leonardo Pisano dual quaternions combining Leonardo Pisano numbers and dual quaternions. Then we examine some fundamental properties and identities of the Leonardo Pisano dual quaternions, such as recurrence relations, generating function, summing formulas, Binet's formula, Cassini and Catalan's identities.

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1. Introduction

"Real quaternions" or simply "quaternions" are a natural extension of complex numbers and they are defined by the following set:

$$\mathbf{H}=\{Q=w_0+w_1i+w_2j+w_3k : w_n\in\mathbb{R}, i^2=j^2=k^2=ijk=-1\}$$

where $1,i,j,k$ are real quaternionic units. It can be seen from the relations of quaternionic units that quaternion algebra is associative but not commutative [9]. For any real quaternion Q, $S_Q=w_0$ is the scalar(real) part and $V_Q=w_1i+w_2j+w_3k$ is the vector part of Q. Moreover, addition of two quaternion is defined component-wise. For the quaternions $Q=S_Q+V_Q$ and $P=S_P+V_P$ addition is defined as $Q+P=(S_Q+S_P)+(V_Q+V_P)$. In addition to this, multiplication of real quaternions can be defined with the rules of quaternionic units. For a quaternion Q, conjugate $\overline{Q}$ is defined by $\overline{Q}=w_0-w_1i-w_2j-w_3k$ and norm $\|Q\|$ is defined by

$$\|Q\|=Q\overline{Q}=w_0^2+w_1^2+w_2^2+w_3^2,\quad \|Q\|\in\mathbb{R}. \tag{1.1}$$

Further information about quaternions can be found in [12, 13, 20].

There are different extensions of quaternions in the literature. Some of them are obtained by changing their coefficients. Some others obtained by chaining the role of quaternionic units. For example, dual quaternions are defined by $i^2=j^2=k^2=ijk=0$, [15]. This type of quaternions are defined as follows:

$$\mathbf{H_D}=\{Q=w_0+w_1i+w_2j+w_3k : w_n\in\mathbb{R}, i^2=j^2=k^2=ijk=0\} \tag{1.2}$$

Further informations on dual quaternions can be found in [4, 6].

In the literature, there are many types of number sequences like Fibonacci, Jacobsthal, Pell, Narayana, Leonardo Pisano, Fubini, and Eulerian numbers, [5, 17]. The authors have studied Leonardo Pisano numbers and also they give some important features about this recurrence relations in [3]. For $n\geq 2$ and initial conditions $Le_0=Le_1=1$, the recurrence relation below defines the Leonardo Pisano numbers

$$Le_n=Le_{n-1}+Le_{n-2}+1 \tag{1.3}$$

Furthermore, another equation related with the Leonardo Pisano number is

$$Le_{n+1}=2Le_n-Le_{n-2},\quad n\geq 2. \tag{1.4}$$

The following equation can be given as a connection of Leonardo Pisano numbers and Fibonacci numbers

$$Le_n = 2F_{n+1} - 1, \quad n \geq 0 \tag{1.5}$$

and also the Binet's formula of Leonardo Pisano numbers can be given as

$$Le_n = \frac{2\varphi^{n+1} - 2\omega^{n+1} - \varphi + \omega}{\varphi - \omega}$$

where φ and ω are the roots of the characteristic equation $\lambda^3 - 2\lambda^2 + 1 = 0$. In [3] the authors have defined Catalan, Cassini, and d'Ocagne's identities. Generalized Leonardo Pisano numbers have been defined in [18]. In [2], incomplete Leonardo Pisano numbers have been given, and also in [14], Leonardo Pisano hybrinomials and their fundamental properties have been studied.

In [21], the authors introduced nth Fibonacci dual quaternion (FQ_n) and Lucas dual quaternion (LQ_n) as following:

$$FQ_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} \tag{1.6}$$

and

$$LQ_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3} \tag{1.7}$$

where F_n and L_n are the nth Fibonacci and Lucas numbers respectively and also $1, i, j, k$ are dual quaternion units.

Further properties about Fibonacci (FQ_n) and Lucas (LQ_n) quaternions can be found in [8, 11, 21]. Several different types of quaternions and their properties can be found in [7, 16, 19, 22].

Some important identities about Leonardo Pisano, Fibonacci, and Lucas numbers which we use in this paper can be listed as follows:

$$Le_n = \frac{2}{5}(L_n + L_{n+2}) - 1, \qquad [3] \tag{1.8}$$

$$Le_{n+3} = \frac{L_{n+1} + L_{n+7}}{5} - 1, \qquad [3] \tag{1.9}$$

$$Le_n = L_{n+2} - F_{n+2} - 1, \qquad [3] \tag{1.10}$$

$$\sum_{t=0}^{n}(F_t + Le_t) = F_{n+2} + Le_{n+2} - (n+3), \qquad [3] \tag{1.11}$$

$$Le_{n+m} + (-1)^m Le_{n-m} = L_m(Le_n + 1) - 1 - (-1)^m, \qquad [1] \tag{1.12}$$

$$FQ_{m+n} - FQ_{m-n} = L_n FQ_m, \qquad [11] \tag{1.13}$$

$$FQ_n^2 - FQ_{n-1}FQ_{n+1} = (-1)^{n+1}(2FQ_1 - 3k), \qquad [11] \tag{1.14}$$

2. Leonardo Pisano Dual Quaternions

In this original section, we will define Leonardo Pisano dual quaternions and present some identities about them.

Definition 2.1: The nth Leonardo Pisano dual quaternion is denoted by LeQ_n and defined as *follows*

$$LeQ_n = Le_n + iLe_{n+1} + jLe_{n+2} + kLe_{n+3} \tag{2.1}$$

where Le_n is nth Leonardo Pisano number and $1, i, j, k$ are dual quaternionic units which have been defined in (1.2). Note that $LeQ_0 = 1 + i + 3j + 5k,\ LeQ_1 = 1 + 3i + 5j + 9k.$

Theorem 2.2: *The following equalities are hold for Leonardo Pisano dual quaternions:*

(i) $LeQ_n = LeQ_{n-1} + LeQ_{n-2} + q,$

(ii) $LeQ_{n+1} = 2LeQ_n - LeQ_{n-2}.$

where $q = 1 + i + j + k.$

Proof: Let LeQ_n be a Leonardo Pisano dual quaternion as in (2.1), then

(i) Taking into account the equation (1.3)

$$\begin{aligned} LeQ_n &= Le_n + iLe_{n+1} + jLe_{n+2} + kLe_{n+3} \\ &= (Le_{n-1} + Le_{n-2} + 1) + i(Le_n + Le_{n-1} + 1) \\ &\quad + j(Le_{n+1} + Le_n + 1) + k(Le_{n+2} + Le_{n+1} + 1) \\ &= LeQ_{n-1} + LeQ_{n-2} + q. \end{aligned}$$

(ii) Using the equation (1.4) we can obtain the desiring result.

Using the definition (2.1), some initial Leonardo Pisano dual quaternions can be written as follows:

$$\begin{aligned} LeQ_0 &= 1 + i + 3j + 5k, \\ LeQ_1 &= 1 + 3i + 5k + 9k, \\ LeQ_2 &= 3 + 5i + 9j + 15k, \\ &\vdots \qquad\qquad \vdots \end{aligned}$$

Definition 2.3: Let $LeQ_n = Le_n + iLe_{n+1} + jLe_{n+2} + kLe_{n+3}$ and $LeQ_m = Le_m + iLe_{m+1} + jLe_{m+2} + kLe_{m+3}$ be two Leonardo Pisano dual quaternions and λ

be a scalar. Addition(subtraction), multiplication, and multiplication with a scalar are defined as follows

$$\begin{aligned} LeQ_n \mp LeQ_m &= (Le_n \mp Le_m) + i(Le_{n+1} \mp Le_{m+1}) \\ &\quad + j(Le_{n+2} \mp Le_{m+2}) + k(Le_{n+3} \mp Le_{m+3}), \\ LeQ_n LeQ_m &= (L \quad e_n Le_m) + i(Le_n Le_{m+1} + Le_{n+1} Le_m) \\ &\quad + j(Le_n Le_{m+2} + Le_{n+2} Le_m) + k(Le_n Le_{m+3} + Le_{n+3} Le_m), \\ \lambda LeQ_n &= \lambda Le_n + \quad i\lambda Le_{n+1} + j\lambda Le_{n+2} + k\lambda Le_{n+3}. \end{aligned}$$

Theorem 2.4 : *Let LQ_n be nth Lucas dual quaternion and n be a positive integer. Then the following identity holds:*

$$LeQ_{n-1} + LeQ_{n+1} = 2LQ_{n+1} - 2q$$

where $q = 1 + i + j + k$.

Proof: We have

$$\begin{aligned} & LeQ_{n-1} + LeQ_{n+1} \\ &= (Le_{n-1} + Le_{n+1}) + i(Le_n + Le_{n+2}) + j(Le_{n+1} + Le_{n+3}) + k(Le_{n+2} + Le_{n+4}) \\ &= [2(F_n + F_{n+2}) - 2] + i[(2F_{n+1} + F_{n+3}) - 2] + j[(2F_{n+2} + F_{n+4}) - 2] \\ &\quad + k[(2F_{n+3} + F_{n+5}) - 2] \\ &= (2L_{n+1} - 2) + i(2L_{n+2} - 2) + j(2L_{n+3} - 2) + k(2L_{n+4} - 2) \\ &= 2LQ_{n+1} - 2q \end{aligned}$$

where F_n, L_n, and LQ_n is the nth Fibonacci number, Lucas number and Lucas dual quaternion respectively.

Theorem 2.5: *If n is a positive integer and $n \geq 2$, then the following identity is true:*

$$LeQ_{n+2} - LeQ_{n-2} = 2(FQ_{n+3} - FQ_{n-1})$$

where FQ_n is nth Fibonacci dual quaternion.

Proof: We have

$$\begin{aligned} LeQ_{n+2} - LeQ_{n-2} = &\quad (Le_{n+2} - Le_{n-2}) + i(Le_{n+3} - Le_{n-1}) \\ &+ j(Le_{n+4} - Le_n) + k(Le_{n+5} - Le_{n+1}) \\ = &\quad 2(F_{n+3} - F_{n-1}) + 2i(F_{n+4} - F_n) \end{aligned}$$

$$\begin{aligned} &+2j(F_{n+5}-F_{n+1})+2k(F_{n+6}-F_{n+2}) \\ = \;& 2(FQ_{n+3}-FQ_{n-1}) \end{aligned}$$

where F_n and FQ_n are nth Fibonacci number and nth Fibonacci dual quaternion, respectively.

Definition 2.6: For a Leonardo Pisano dual quaternion LeQ_n, the conjugate is defined as

$$\overline{LeQ_n} = Le_n - iLeQ_{n+1} - jLeQ_{n+2} - kLeQ_{n+3}.$$

Theorem 2.7: *Following equations are valid for Leonardo Pisano dual quaternions for* $n \geq 0$

i. $LeQ_n \overline{LeQ_n} = Le_n^2$,

ii. $LeQ_n + \overline{LeQ_n} = 2Le_n$

where LeQ_n *is* nth *Leonardo Pisano dual quaternion and* Le_n *is* nth *Leonardo Pisano number.*

Proof: Straightforward.

Theorem 2.8: *Let n be a non-negative integer. The following identity holds:*

$$LeQ_n - iLeQ_{n+1} - jLeQ_{n+2} - kLeQ_{n+3} = Le_n$$

where Le_n *is* nth *Leonardo Pisano number.*

Proof: By using the basic operations we can obtain desired result as follows:

$$\begin{aligned} &LeQ_n - iLeQ_{n+1} - jLeQ_{n+2} - kLeQ_{n+3} \\ &= Le_n + iLe_{n+1} + jLe_{n+2} + kLe_{n+3} \\ &\quad - i(Le_{n+1} + iLe_{n+2} + jLe_{n+3} + kLe_{n+4}) \\ &\quad - j(Le_{n+2} + iLe_{n+3} + jLe_{n+4} + kLe_{n+5}) \\ &\quad - k(Le_{n+3} + iLe_{n+4} + jLe_{n+5} + kLe_{n+6}) = Le_n. \end{aligned}$$

Now, we introduce the generating function for Leonardo Pisano dual quaternions. Using the definition of generating functions.

Definition 2.9: The generating function of the Leonardo Pisano dual quaternions is

$$g_{LeQ}(t)=\sum_{p=0}^{\infty} t^p LeQ_p. \tag{2.2}$$

Theorem 2.10: *The generating function of the Leonardo Pisano dual quaternions is*

$$g_{LeQ}(t)=\frac{(1+i+3j+5k)+t(-1+i-j-k)+t^2(1-i-j-3k)}{1-2t+t^3}$$

such that $1-2t+t^3\neq 0$.

Proof: If we sum up the following equations

$$\begin{aligned}
g_{LeQ}(t)&=LeQ_0+tLeQ_1+t^2LeQ_2+t^3LeQ_3+t^4LeQ_4+\dots\\
-2tg_{LeQ}(t)&=-2tLeQ_0-2t^2LeQ_1-2t^3LeQ_2-2t^4LeQ_3-2t^5LeQ_4+\dots\\
t^3g_{LeQ}(t)&=t^3LeQ_0+t^4LeQ_1+t^5LeQ_2+t^6LeQ_3+t^7LeQ_4+\dots
\end{aligned}$$

and using (2.2) we get

$$g_{LeQ}(t)=\frac{(1+i+3j+5k)+t(-1+i-j-k)+t^2(1-i-j-3k)}{1-2t+t^3}.$$

Theorem 2.11: *Let* LeQ_n *be* nth *Leonardo Pisano dual quaternion and* FQ_n *be* nth *Fibonacci dual quaternion. The relation between Leonardo Pisano dual quaternions and Fibonacci dual quaternions can be given as following:*

$$LeQ_n=2FQ_{n+1}-q \tag{2.3}$$

where $q=1+i+j+k$.

Proof: Desiring equation can be easily shown by using ().

Theorem 2.12: *Let n and r be two positive integers, then the following identities are true:*

(i) $LeQ_{n+1}FQ_{n+1}-LeQ_nFQ_n=(LeQ_{n-1}+q)FQ_n-LeQ_{n+1}FQ_{n-1}$.

(ii) If n is odd, then

$$LeQ^2_{m+n} - LeQ^2_{m-n} = 4(FQ^2_{m+n+1} - FQ^2_{m-n+1}) - 2L_n FQ_{m+1} q - 2qL_n FQ_{m+1}.$$

(iii) $LeQ_{n+r} + (-1)^r LeQ_{n-r} = \begin{cases} 2(L_r FQ_{n+1} - q) & r \text{ is even} \\ 2L_r FQ_{n+1} & r \text{ is odd} \end{cases}$

(iv) $FQ_r LeQ_{r-1} - FQ_{r-1} LeQ_r = 2(-1)^{r+1}(2FQ_1 - 3k) - FQ_{r-2} q.$

Proof:

(i) Using the theorem (2.2)

$$\begin{aligned} LeQ_{n+1}FQ_{n+1} - LeQ_n FQ_n &= LeQ_{n+1}(FQ_n + FQ_{n-1}) - LeQ_n FQ_n \\ &= (LeQ_{n+1} - LeQ_n)FQ_n + LeQ_{n+1}FQ_{n-1} \\ &= (LeQ_{n-1} + q)FQ_n + LeQ_{n+1}FQ_{n-1}. \end{aligned}$$

(ii) Let n be an odd integer. With using the identity (1.13) and (2.3) we obtain

$$\begin{aligned} &LeQ^2_{m+n} - LeQ^2_{m-n} \\ &= (2FQ_{m+n+1} - q)^2 - (2FQ_{m-n+1} - q)^2 \\ &= 4FQ^2_{m+n+1} - 2FQ_{m+n+1}q - 2qFQ_{m+n+1} + q^2 \\ &\quad -(4FQ^2_{m-n+1} - 2FQ_{m-n+1}q - 2qFQ_{m-n+1} + q^2) \\ &= 4(FQ^2_{m+n+1} - FQ^2_{m-n+1}) - 2(FQ_{m+n+1} - FQ_{m-n+1})q \\ &\quad -2q(FQ_{m+n+1} - FQ_{m-n+1}) \\ &= 4(FQ^2_{m+n+1} - FQ^2_{m-n+1}) - 2L_n FQ_{m+1} q - 2qL_n FQ_{m+1}. \end{aligned}$$

(iii) Using (2.3)

$$\begin{aligned} LeQ_{n+r} + (-1)^r LeQ_{n-r} &= (2FQ_{n+r+1} - q) + (-1)^r (2FQ_{n-r+1} - q) \\ &= 2(FQ_{n+r+1} + (-1)^r FQ_{n-r+1}) - q(1 + (-1)^r) \end{aligned}$$

when if $n+1=m,$ then we get

$$\begin{aligned} LeQ_{n+r} + (-1)^r LeQ_{n-r} &= 2(FQ_{m+r} + (-1)^r FQ_{m-r}) - q(1 + (-1)^r) \\ &= 2L_r FQ_m - q(1 + (-1)^r) \\ &= 2L_r FQ_{n+1} - q(1 + (-1)^r). \end{aligned}$$

(iv) Using (1.14) and (2.3) we can obtain

$$\begin{aligned} FQ_r LeQ_{r-1} - FQ_{r-1} LeQ_r &= FQ_r(2FQ_r - q) - FQ_{r-1}(2FQ_{r+1} - q) \\ &= 2(FQ_r^2 - FQ_{r-1}FQ_{r+1}) - (FQ_r - FQ_{r-1})q \\ &= 2(-1)^{r+1}(2FQ_1 - 3k) - FQ_{r-2}\, q. \end{aligned}$$

Theorem 2.13: *If n and m are non-negative integers and $n \geq m$, then the following identities hold:*

i. $LeQ_{n+m} + (-1)^m LeQ_{n-m} = L_m(LeQ_n + q) - (1 + (-1)^m)q$

ii. $LeQ_{n+m} - (-1)^m LeQ_{n-m} = L_{n+1}(LeQ_{m-1} + q) + (-1)^m(1 - i + j - k) - q$

where $q = 1 + i + j + k$ and L_m is mth Lucas number.

Proof:

i. Using the equation (1.12), we obtain

$$\begin{aligned} &LeQ_{n+m} + (-1)^m LeQ_{n-m} \\ &= (Le_{n+m} + (-1)^m Le_{n-m}) + i(Le_{n+m+1} + (-1)^m Le_{n-m+1}) \\ &\quad + j(Le_{n+m+2} + (-1)^m Le_{n-m+2}) + k(Le_{n+m+3} + (-1)^m Le_{n-m+3}) \\ &= (L_m(Le_n + 1) - 1 - (-1)^m) + i(L_m(Le_{n+1} + 1) - 1 - (-1)^m) \\ &\quad + j(L_m(Le_{n+2} + 1) - 1 - (-1)^m) + k(L_m(Le_{n+3} + 1) - 1 - (-1)^m) \\ &= L_m(LeQ_n + q) - (1 + (-1)^m)q. \end{aligned}$$

ii. It can be proved similarly to **i**.

Theorem 2.14 (Binet's formula): *Let LeQ_n be any Leonardo Pisano dual quaternion, then Binet's formula is*

$$LeQ_n = \frac{2(\varphi^* \varphi^{n+1} - \omega^* \omega^{n+1})}{\varphi - \omega} - q$$

where $q = 1 + i + j + k$, $\varphi^* = 1 + i\varphi + j\varphi^2 + k\varphi^3$, $\omega^* = 1 + i\omega + j\omega^2 + k\omega^3$, and φ, ω are roots of the equation $x^3 - 2x^2 + 1 = 0$.

Proof: Using Binet's formula for Leonardo Pisano numbers, we get

$$LeQ_n = 2\left(\frac{\varphi^{n+1} - \omega^{n+1}}{\varphi - \omega}\right) + 2i\left(\frac{\varphi^{n+2} - \omega^{n+2}}{\varphi - \omega}\right)$$

$$+2j\left(\frac{\varphi^{n+3}-\omega^{n+3}}{\varphi-\omega}\right)+2k\left(\frac{\varphi^{n+4}-\omega^{n+4}}{\varphi-\omega}\right)-q$$
$$=\frac{2\varphi^{n+1}(1+i\varphi+j\varphi^2+k\varphi^3)}{\varphi-\omega}-\frac{2\omega^{n+1}(1+i\omega+j\omega^2+k\omega^3)}{\varphi-\omega}-q$$
$$=\frac{2(\varphi^*\varphi^{n+1}-\omega^*\omega^{n+1})}{\varphi-\omega}-q.$$

Now we give some sum formulas for Leonardo Pisano dual quaternions with the help of the formulas for the sum of Leonardo Pisano numbers.

Theorem 2.15 : *For* $n \geq 0$

i. $\sum_{k=0}^{n} LeQ_k = LeQ_{n+2}-(n+2)q-(2i+4j+8k)$,
ii. $\sum_{k=0}^{n} LeQ_{2k} = LeQ_{2n+1}-nq-(2i+2j+4k)$,
iii. $\sum_{k=0}^{n} LeQ_{2k+1} = LeQ_{2n+2}-(n+2)q-(2j+4k)$

where $q=1+i+j+k$.

Proof: The proof of these equations can be obtained by using the sum formulas for Leonardo Pisano numbers.

Theorem 2.16 (Catalan's Identity): *Let n and r be non-negative integers and* $n \geq r$, *then the following identity satisfies*

$$LeQ_n^2-LeQ_{n-r}LeQ_{n+r}=(-1)^{n+r}4F_r((2+i+3j+4k)F_r-2jL_r$$
$$+(LeQ_n+LeQ_{n-r})q+q(LeQ_n+LeQ_{n+r})+4q^2$$

where $q=1+i+j+k$.

Proof: If we use the equation (2.3),

$$LeQ_n^2-LeQ_{n-r}\quad LeQ_{n+r}=(2FQ_{n+1}-q)^2-(2FQ_{n-r+1}-q)(2FQ_{n+r+1}-q)$$
$$=4(FQ_{n+1}^2-FQ_{n-r+1}FQ_{n+r+1})$$
$$+(2FQ_{n+1}+2FQ_{n-r+1})q+q(2FQ_{n+1}+2FQ_{n+r+1})$$
$$=4(FQ_{n+1}^2-FQ_{n-r+1}FQ_{n+r+1})$$
$$+(LeQ_n+LeQ_{n-r}+2q)q+q(LeQ_n+LeQ_{n+r}+2q)$$
$$=4(FQ_{n+1}^2-FQ_{n-r+1}FQ_{n+r+1})$$
$$+(LeQ_n+LeQ_{n-r})q+q(LeQ_n+LeQ_{n+r})+4q^2$$
$$=(-1)^{n+r}4F_r((2+i+3j+4k)F_r-2jL_r$$
$$+(LeQ_n+LeQ_{n-r})q+q(LeQ_n+LeQ_{n+r})+4q^2$$

where $q=1+i+j+k$.

In the case of $r=1$ in the previous result and taking into account the first item of theorem (2.2), we have the following result.

Corollary 2.17 (Cassini Identity): *Let n be a non-negative integer. The following identity holds*

$$LeQ_n^2 - LeQ_{n-1}LeQ_{n+1} = (-1)^{n+1}4(2+i+j+4k) + LeQ_{n+1}q + qLeQ_{n+2} + 2q^2$$

where $q=1+i+j+k$.

Now, some more identities satisfied by Leonardo Pisano dual quaternions are stated.

Theorem 2.18: *If n is a non-negative integer, then the following identities satisfy:*

i. $LeQ_n = \frac{2}{5}(LQ_n + LQ_{n+2}) - q$,
ii. $LeQ_{n+3} = \frac{LQ_{n+1}+LQ_{n+7}}{5} - q$,
iii. $LeQ_n = LQ_{n+2} - FQ_{n+2} - q$

where $q=1+i+j+k$. LeQ_n *is the nth Leonardo Pisano dual quaternion and* LQ_n *is the nth Lucas quaternion.*

Proof: These can be easily shown by using identities (1.8), (1.9) and (1.10).

Several sum formulas involving different types of dual quaternions are established in the following results.

Theorem 2.19: *Let n be a positive integer. The following identities hold:*

i. $\sum_{t=0}^{n}(FQ_t + LeQ_t) = FQ_{n+2} + LeQ_{n+2} - (n+3)q - (2i+5j+10k)$,
ii. $\sum_{t=0}^{n}(FQ_t + LeQ_t) = FQ_{n+5} - (n+4)q - (2i+5j+10k)$,
iii. $\sum_{t=0}^{n}(LQ_t + LeQ_t) = LQ_{n+2} + LeQ_{n+2} - (n+3)q - (4i+7j+14k)$,
iv. $\sum_{t=0}^{n}(LQ_t + LeQ_t) = 2FQ_{n+3} + LQ_{n+2} - (n+4)q - (4i+7j+14k)$

where $q=1+i+j+k$. *Furthermore,* FQ_n, LeQ_n *and* LQ_n *are the nth Fibonacci quaternion, Leonardo Pisano dual quaternion, and Lucas quaternion respectively.*

Proof: Here we will prove just the first identity. Using the equation (1.11),

$$\begin{aligned}
\sum_{t=0}^{n}(FQ_t+LeQ_t) &= F_0+iF_1+jF_2+kF_3+Le_0+iLe_1+jLe_2+kLe_3 \\
&\quad +F_1+iF_2+jF_3+kF_4+Le_1+iLe_2+jLe_3+kLe_4 \\
&\quad \vdots \\
&\quad +F_n+iF_{n+1}+jF_{n+2}+kF_{n+3}+Le_n+iLe_{n+1}+jLe_{n+2}+kLe_{n+3} \\
&= F_{n+2}+Le_{n+2}-(n+3) \\
&\quad +i(F_{n+3}+Le_{n+3}-(n+4)-F_0-Le_0) \\
&\quad +j(F_{n+4}+Le_{n+4}-(n+5)-F_0-F_1-Le_0-Le_1) \\
&\quad +k(F_{n+5}+Le_{n+5}-(n+6)-F_0-F_1-F_2-Le_0-Le_1-Le_2) \\
&= FQ_{n+2}+LeQ_{n+2}-(n+3)q-(2i+5j+10k).
\end{aligned}$$

Other identities can be proved similarly.

3. Conclusion

In this study, it has been introduced Leonardo Pisano dual quaternions. Then some important identities and properties about the Leonardo Pisano dual quaternions, including the Cassini identity and Binet's formula, have been given and proven.

References

[1] Alp Y, Kocer EG . Some properties of Leonardo numbers, *Konuralp Journal of Mathematics*, 9(1) : 183-9 (2021).

[2] Catarino. P., Borges A A note on incomplete Leonardo numbers. *Integers* :20 (2020).

[3] Catarino, P., Borges A.: On Leonardo numbers. *Acta Math Univ Comenianae*, 89(1) : 75-86 (2020).

[4] Dağdeviren, A., Yüce, S.: Dual quaternions and dual quaternionic curves. *Filomat*, 33, 1037-1046 (2019).

[5] Dağdeviren, A., Kuruz, F.: On the Horadam hybrid quaternions. *Advanced Studies: Euro-Tbilisi Mathematical Journal* (10), pp. 293-306 (2022).

[6] Ercan Z., Yüce S.: On properties of the dual quaternions. *European Journal of Pure and Applied Mathematics* 4(2) 142-146 (2011).

[7] Halici, S.: On complex Fibonacci quaternions. *Adv. Appl. Clifford Algebras* 23, 105-112 (2013).

[8] Halici, S.: On fibonacci quaternions. *Adv. Appl. Clifford Algebras*, 22(2), 321-327 (2012) https://doi.org/10.1007/s00006-011-0317-1.

[9] Hamilton, W. R.: Elements of Quaternions. Longmans, Green and Co., London, (1866).

[10] Horadam, A.F., Complex Fibonacci Numbers and Fibonacci Quaternions, *Amer. Math*. Monthly, 70, 289-291 (1963).

[11] Iyer, M. R., Some results of Fibonacci quaternions, *The Fibonacci Quarterly*, 15(4), 225-230, (1969).

[12] Kantor, I.L., Solodnikov, A.S.: Hypercomplex Numbers, an Elementary Introduction to algebras. Springer, New York, (1989).

[13] Kuipers, J.B.: Quaternions and Rotation Sequences. Princeton University Press, Princeton, New Jersey (1999).

[14] Kürüz, F., Dağdeviren, A., Catarino, P.: On Leonardo Pisano Hybrinomials. *Mathematics* 9.22, 2923, (2021).

[15] Majernik, V.: Quaternion formulation of the Galilean space-time transformation. *Acta Phy. Slovaca.* 56(1), 9-14 (2006).

[16] Nurkan, S.K., Güven, I.A.: Dual Fibonacci quaternions. Adv. Appl. Clifford Algebras 25(2), 403-414 (2015).

[17] Qi, F.: Determinantal expressions and recurrence relations for Fubini and Eulerian polynomials, *Journal of Interdisciplinary Mathematics* 22(3), 317-335 (2019).

[18] Shannon, A. G.: A note on generalized Leonardo numbers. *Notes Number Theory Discrete Math;* 25(3) : 97-101 (2019). doi: 10.7546/nntdm.2019.25.3.97-101.

[19] Szynal-Liana A., Wloch I.: The Pell quaternions and the Pell octonions. *Adv. Appl. Clifford Algebras*. 26, 1, 435-440 (2016), doi:10.1007/s00006-015-0570-9.

[20] Ward, J.P.: Quaternions and Cayley Numbers: Algebra and Applications. *In: Mathematics and Its Applications*, vol. 403, Kluwer, Dordrecht (1997).

[21] Yüce, S., Aydın F.T.: A new aspect of dual Fibonacci quaternions. *Advances in applied Clifford algebras*, 26.2, 873-884, (2016).

[22] Zhang, F.: Quaternions and matrices of quaternions. *Linear algebra and its applications*, 251, 21-57 (1997).

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