Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-B, pp. 1217–1224
DOI : 10.47974/JIM-2210

Investigating of the attributes for a novel type of extended soft quasinormal for normal operator in Hilbert space

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Abstract

Through our study of soft-liner operator in Hilbert space we discovered a new type, which is a Novel Type of Extended Soft quasinormal for Normal Operator in Hilbert Space In this paper we present some definitions and characteristics related to soft Ғ̂- quasi Ķ̂-normal operator Furthermore, we explain the different properties of this operator as well as direct addition and effective multiplication.

Subject Classification: *Primary 93A30, Secondary 49K15.*

Keywords: *Soft normal operator, Soft quasinormal operator, Soft Hilbert space.*

1. Introduction

Often, there is ambiguity in the truth resulting from uncertainty, and for this reason, we face problems in various fields, including physics, medicine, engineering, economics, social sciences, and computer science In 1999,

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Molodtsov [1] introduced soft set concept and it was considered extension for set theory, allowing of easy application for this theory without restrictions to address and solve problems arising from uncertainty that traditional methods cannot resolve "Then, others presented, including Maji et al [2-3] many operations on soft sets through the application of soft set theory. A lot of these works in this direction have been found, while the definition of reducing the parameters of soft sets emerged, and we have seen some works in this field [4, 5, 6]"Presented by Chen [7], this work also studied soft real numbers and soft composite numbers [8-9], soft architectural spaces, softmetric space [10-11], and softlinear space, along with softHilbert spaces, while examining some of their properties. Furthermore, through the advantage that the operator possesses, it ranks among the top in functional analysis, which is considered a generalization of the concept of a function. In 1953, Brown [12] for the first time linked the new concept with the operator theory, where the operator is called quasi-normal operator. The researchers studied this field and presented some specific features of quasi-normal operators. Research in this direction has continued until 2010, witnessing significant developments in various types of quasi-operators. Through the theory of soft inner product, which was further expanded in 2013, the concept of soft inner product was defined. [13] Some properties and features related to the concept of soft inner product were studied, and this theory was expanded through operators on soft inner product spaces and soft linear operators.[14-15, 18-23] In this context, a new formulation of the quasi-normal operator in Hilbert space will be presented, along with an examination of some properties and theories that are considered essential conditions for this operator.

2. Elementary principles in soft theory

Definition 2.1: [1] Assume that the universal set X should have the parameters make up the set E, such that $A \subseteq E$, soft sets over X is (f , A), when $f : A \rightarrow p(x)$.

Definitions 2.2: [9] If A set for parameters & C set for complex numbers. Let $p(C)$ family for all nonempty bounded subset for C. the map $f: A \rightarrow p(C)$ is then referred to as a soft complex set. The symbol of its (f, A).

Definitions 2.3: [13] If A is parameter set & X vector space over field F.Allow a delicate set of G over X. If $G(\beta)$ vector subspace for $X \, \forall \beta \in A$, then G is described as a soft vector space of X over F.

Definitions 2.4: **[17]** Let $SE\big(\tilde{V} \big)$ the mapping be a soft vector space. $\langle .,. \rangle : SE\big(\tilde{V} \big) \times SE\big(\tilde{V} \big) \longrightarrow R(E)^{*}$ called soft inner product on $SE\big(\tilde{V} \big)$ iff it satisfied following conditions $\forall\, x_{e}, y_{e}, z_{e} \in SE\big(\tilde{V} \big)$ and for every soft real number $\widetilde{\beta}$:

1. $\langle x_{e} , x_{e} \rangle \geq \tilde{0}$ and $\langle x_{e} , x_{e} \rangle = \tilde{0}$ if and only if $x_{e} = \tilde{\Theta}$

2. $\langle y_e, x_e \rangle = \langle x_e, y_e \rangle$
3. $\langle x_e, \widetilde{\beta}\ y_e \rangle = \widetilde{\beta} \langle x_e, y_e \rangle = \langle \widetilde{\beta}\, x_e, y_e \rangle$
4. $\langle x_e, z_e \rangle + \langle y_e, z_e \rangle = \langle x_e + y_e, z_e \rangle$, the triple $\left(SE(\tilde{V}), \langle . \rangle, E \right)$ is called soft inner product space.

Definition 2.5: [16] The soft complete inner product space $\left(SE(\tilde{V}), \langle .,. \rangle\right)$ is also known as a soft Hilbert space ($SH -$ space) & is denote as the sign $SE(H)$.

Definition 2.6: [14] If $\hat{\text{F}} : SE(H) \rightarrow SE(H)$ soft operator in $SH -$ space H, then $\hat{\text{F}}$ is namely soft linear operators ($SL -$ operator), then $\hat{\text{F}} \in L(H)$ if for every $x_{e1}, y_{e2} \in H$ satisfy $\hat{\text{F}}\ (\beta\ x_{e1} + \beta\, y_{e2}) = \beta\, \hat{\text{F}}\ (x_{e1}) + \beta\, \hat{\text{F}}\ (y_{e2})$, and ($\beta$, β are $S -$ scalars).

Definition 2.7: [17] Let $\hat{\text{F}} : SE(H) \rightarrow SE(H)$ be ($S -$ operator) is soft bounded operator ($SB -$ operator), $\exists m \in R^{+}(A)$ s.t $\| \Psi\Upsilon(x_{e1})\| \leq m \, \|(x_{e1})\|$, $\forall x_{e1} \in SE(H)$.

3. Proposed generalized soft operator and its traits

Definition 3.1: Suppose $\hat{\text{F}} : SE(H) \rightarrow SE(H)$ be soft a linear operator with bounded so $\hat{\text{F}}$ claimed as soft $\hat{\text{F}} -$ quasi $\hat{\text{K}} -$normal operator if $\hat{\text{F}}^{*} \left(\left(\hat{\text{F}}^{*} \right)^{v} \hat{\text{F}} \right) = \hat{\text{K}}\left(\left(\hat{\text{F}}^{*} \right)^{v} \hat{\text{F}} \right) \hat{\text{F}}^{*}$, where $v \in Z^{+}$ and $\hat{\text{K}} : SE(H) \rightarrow SE(H)$is soft bounded operator.

To illustrate this definition, one can see the operator matrix, $\hat{\text{F}} = \left[\underline{0}\ \underline{0}\ \underline{1}\ \underline{0} \right]$ to be soft $\hat{\text{F}} -$ quasi $\hat{\text{K}} -$normal operator where $v > 2$, but not satisfy in case $v = 1$.

These proofs now provide certain properties of the soft $\hat{\text{F}} -$ quasi $\hat{\text{K}}$ $-$normal operator.

Proposition 3.2: Let $\hat{\text{F}} : SE(H) \rightarrow SE(H)$ is soft $\hat{\text{F}} -$ quasi $\hat{\text{K}} -$normal operator, then i) $\alpha\, \hat{\text{F}}$ is soft $\hat{\text{F}} -$ quasi $\hat{\text{K}} -$normal operator where $\alpha \in R(A)$.

ii) $\frac{\hat{\text{F}}}{\text{M}}$ is soft $\hat{\text{F}} -$ quasi $\hat{\text{K}} -$normal operator where $\widehat{\text{MM}}$ is closed sub space

Proof:

i) $$\left(\beta\, \hat{\text{F}}\right)^{*} \left(\left(\beta\, \hat{\text{F}}\right)^{*}\right)^{v} \left(\beta\, \hat{\text{F}}\right) = \beta\, \hat{\text{F}}^{*} \left(\beta^{v} \left(\hat{\text{F}}^{*}\right)^{v}\right) \left(\beta\, \hat{\text{F}}\right)$$
$$= \beta\, \beta^{v}\, \beta\, \hat{\text{F}}^{*} \left(\hat{\text{F}}^{*}\right)^{v} \hat{\text{F}}$$
$$= \beta^{v}\, \beta\, \beta\, \hat{\text{K}} \left(\hat{\text{F}}^{*}\right)^{v} \hat{\text{F}}\, \hat{\text{F}}^{*}$$
$$= \hat{\text{K}}\left(\left(\beta\, \hat{\text{F}}\right)^{*}\right)^{v} \left(\beta \hat{\text{F}}\right) \left(\beta\, \hat{\text{F}}\right)^{*}$$

Therefore $\beta\, \hat{\text{F}}$ is soft $\hat{\text{F}} -$ quasi $\hat{\text{K}} -$normal operator.

ii) $\left(\frac{\hat{F}}{\widehat{M}}\right)^{*}\left(\left(\left(\frac{\hat{F}}{\widehat{M}}\right)^{*}\right)^{v}\left(\frac{\hat{F}}{\widehat{M}}\right)\right)=\left(\frac{\hat{F}^{*}}{\widehat{M}}\right)\left(\frac{\left(\hat{F}^{*}\right)^{v}\hat{F}}{\widehat{M}}\right)$

$$=\left(\frac{\hat{F}^{*}\left(\hat{F}^{*}\right)^{v}\hat{F}}{\widehat{M}}\right)$$

$$=\left(\frac{\widehat{K}\left(\hat{F}^{*}\right)^{v}\hat{F}\,\hat{F}^{*}}{\widehat{M}}\right)$$

$$=\widehat{K}\left(\left(\left(\frac{\hat{F}}{\widehat{M}}\right)^{*}\right)^{v}\left(\frac{\hat{F}}{\widehat{M}}\right)\right)\left(\frac{\hat{F}}{\widehat{M}}\right)^{*}$$

So, we will obtain, exactly $\frac{\hat{F}}{\widehat{M}}$ is soft operator of the type soft $\hat{F}-$ quasi $\widehat{K}$

Proposition 3.3: Every soft self adjoint operator is soft $\hat{F}-$ quasi $\widehat{K}-$normal operator.

Proof: $$\hat{F}^{*}\left(\left(\hat{F}^{*}\right)^{v}\hat{F}\right)=\hat{F}\left(\hat{F}^{v}\hat{F}\right)=\hat{F}^{v+2} \quad (1)$$

$\left(\left(\hat{F}^{*}\right)^{v}\hat{F}\right)\hat{F}^{*}=\left(\hat{F}^{v}\right)\hat{F}\hat{F}=\hat{F}^{v+2}$ ... (2), so, we get (1) = (2),

Therefore; soft $\hat{F}-$ quasi $\widehat{K}-$normal operator.

Corollary 3.4: *Given any soft operator* $\hat{F}$ *on Hilbert space* $SE(H)$, $\left(\hat{F}+\hat{F}^{*}\right)$ *also* $\hat{F}\hat{F}^{*}$, $\hat{F}^{*}\hat{F}$, $I+\hat{F}^{*}\hat{F}$ *and* $I+\hat{F}\hat{F}^{*}$ *are soft* $\hat{F}-$ *quasi* $\widehat{K}-$*normal operators.*

Remark 3.5: Given tow soft $\hat{F}-$ quasi $\widehat{K}-$normal operator, $\hat{F}_1$ and $\hat{F}_2$ it can be said that $\hat{F}_1+\hat{F}_2$ is not always an soft $\hat{F}-$ quasi $\widehat{K}-$normal operator. Take at the following instance to demonstrate that.

Example 3.6: If $\hat{F}_1=[1\ 0\ 0\ -1]$ and $\hat{F}_2=[1\ 2\ 3\ 1]$, $\widehat{K}=[1\ 2\ 3\ 4]$ soft $\hat{F}-$ quasi normal operators. If n = 2, then

$$\left(\hat{F}_1+\hat{F}_2\right)^{*}\left(\left(\left(\hat{F}_1+\hat{F}_2\right)^{*}\right)^{2}\left(\hat{F}_1+\hat{F}_2\right)\right)=[2\ 1\ 1\ 2] \quad (1)$$

$$\left(\widehat{K}\left(\left(\hat{F}_1+\hat{F}_2\right)^{*}\right)^{2+\left(\hat{F}_1+\hat{F}_2\right)}\right)\left(\hat{F}_1+\hat{F}_2\right)^{*}=[1\ 2\ 2\ 2] \quad (2)$$

Thus we can get $1\neq 2$, then $\hat{F}_1+\hat{F}_2$ is not soft $\hat{F}-$ quasi $\widehat{K}-$normal operator.

Theorem 3.7: *Assumption* $\hat{F}_1,\hat{F}_2: SE(H)\rightarrow SE(H)$ *be two soft* $\hat{F}-$ *quasi* $\kappa-$*normal operators which is known on in a Hilbert space with condition satisfies* $\left(\hat{F}_1+\hat{F}_2\right)^{*}$ *commutes with* $\left(\sum_{\kappa=1}^{v-1}\ (v\ \kappa)\hat{F}_1^{v-\kappa}\hat{F}_2^{\kappa}\right)^{*}$. *Then* $\left(\hat{F}_1+\hat{F}_2\right)$ *is soft* $\hat{F}-$ *quasi* $\widehat{K}-$*normal operator.*

Proof : By using the definition of is soft $\hat{F}-$ quasi $\hat{K}-$normal operator

$$(\hat{F}_1+\hat{F}_2)^*\left(\left((\hat{F}_1+\hat{F}_2)^*\right)^v(\hat{F}_1+\hat{F}_2)\right)=(\hat{F}_1+\hat{F}_2)^*\left(((\hat{F}_1+\hat{F}_2)^v)^*(\hat{F}_1+\hat{F}_2)\right)$$

$$=(\hat{F}_1+\hat{F}_2)^*((\textstyle\sum_{\kappa=0}^{v}\ (v\ \kappa)\,\hat{F}_1^{v-\kappa}\,\hat{F}_2^{\kappa}\,)^*(\hat{F}_1+\hat{F}_2)),$$

$$=(\hat{F}_1+\hat{F}_2)^*((\hat{F}_1^{v}+\textstyle\sum_{\kappa=1}^{v-1}\ (v\ \kappa)\,\hat{F}_1^{v-\kappa}\,\hat{F}_2^{\kappa}+\hat{F}_2^{v})^*(\hat{F}_1+\hat{F}_2))\text{ we get}$$

$$=(\hat{F}_1+\hat{F}_2)^*((\hat{F}_1^{v}+\textstyle\sum_{\kappa=1}^{v-1}\ (v\ \kappa)\,\hat{F}_1^{v-\kappa}\,\hat{F}_2^{\kappa}+\hat{F}_2^{v})^*(\hat{F}_1+\hat{F}_2))(\hat{F}_1+\hat{F}_2)^*$$

$$(\hat{F}_1^{*v}+(\textstyle\sum_{\kappa=1}^{v-1}\ (v\ \kappa)\,\hat{F}_1^{v-\kappa}\,\hat{F}_2^{\kappa})^*+\hat{F}_2^{*v})(\hat{F}_1+\hat{F}_2)=((\hat{F}_1+\hat{F}_2)^*\hat{F}_1^{*v}+(\hat{F}_1+\hat{F}_2)^*$$

$$(\textstyle\sum_{\kappa=1}^{v-1}\ (v\ \kappa)\,\hat{F}_1^{v-\kappa}\,\hat{F}_2^{\kappa})^*+(\hat{F}_1+\hat{F}_2)^*\,\hat{F}_2^{*v})(\hat{F}_1+\hat{F}_2),$$

Since $(\hat{F}_1+\hat{F}_2)^*$ commuting with $(\sum_{\kappa=1}^{v-1}\ (v\ \kappa)\,\hat{F}_1^{v-\kappa}\,\hat{F}_2^{\kappa})^*$

$$=(\hat{F}_1^{*v}\left(\hat{F}_1^{+\hat{F}_2}\right)^{**+}(\textstyle\sum_{\kappa=1}^{v-1}\ (v\ \kappa)\,\hat{F}_1^{v-\kappa}\,\hat{F}_2^{\kappa})^*\left(\hat{F}_1^{+\hat{F}_2}\right)^*+\hat{F}_2^{*v}(\hat{F}_1+\hat{F}_2)^*)(\hat{F}_1+\hat{F}_2)$$

$$=(\hat{F}_1^{*v}+(\textstyle\sum_{\kappa=1}^{v-1}\ (v\ \kappa)\,\hat{F}_1^{v-\kappa}\,\hat{F}_2^{\kappa})^*+\hat{F}_2^{*v})(\hat{F}_1+\hat{F}_2)^*(\hat{F}_1+\hat{F}_2)$$

$$=(\hat{F}_1^{*v}+(\textstyle\sum_{\kappa=1}^{v-1}\ (v\ \kappa)\,\hat{F}_1^{v-\kappa}\,\hat{F}_2^{\kappa})^*+\hat{F}_2^{*v})(\hat{F}_1+\hat{F}_2)(\hat{F}_1+\hat{F}_2)^*$$

$$=\hat{K}(\left((\hat{F}_1+\hat{F}_2)^*\right)^v(\hat{F}_1+\hat{F}_2))\,(\hat{F}_1+\hat{F}_2)^*$$

Hence $(\hat{F}_1+\hat{F}_2)$ is soft $\hat{F}-$ quasi $\hat{K}-$normal operator.

Remark 3.8: Examine what follows as an example to show that even if $\hat{F}_1$ and $\hat{F}_2$ are two soft $\hat{F}-$ quasi $\hat{K}-$normal operator, $\hat{F}_1\hat{F}_2$ are not necessarily soft $\hat{F}-$ quasi $\hat{K}-$normal operators.

Example 3.9: If $\hat{F}_1=[1\ 0\ 0\ -1\ ]$ and $\hat{F}_2=[1\ 2\ 3\ 1\ ]$, $\hat{K}=[1\ 2\ 3\ 4\ ]$ soft $\hat{F}-$ quasi normal operators. If n = 2, then

$$(\hat{F}_1\hat{F}_2)^*\left(\left((\hat{F}_1\hat{F}_2)^*\right)^2(\hat{F}_1\hat{F}_2)\right)=[1\ 1\ 1\ 1\ ] \tag{1}$$

$$\left(\hat{K}\left((\hat{F}_1\hat{F}_2)^*\right)^2(\hat{F}_1\hat{F}_2)\right)(\hat{F}_1\hat{F}_2)^*=[1\ 0\ 1\ 0\ ] \tag{2}$$

Thus, we can get $1\neq 2$, then $\hat{F}_1\hat{F}_2$ is not soft $\hat{F}-$ quasi $\hat{K}-$normal operator.

Theorem 3.10: *Let* $\hat{F}_1,\hat{F}_2: SE(H)\to SE(H)$ *be two soft* $\hat{F}-$*quasi* $\hat{K}-$*normal operators such that,* $\hat{F}_1\hat{F}_2=\hat{F}_2\hat{F}_1$, $\hat{F}_2^*\hat{F}_1=\hat{F}_1\hat{F}_2^*$ *and* $\left(\hat{F}_1^{*v}\hat{F}_1\right)\hat{F}_1^*\hat{K}_2$

$=\hat{K}_2\left(\hat{F}_1^{*v}\hat{F}_1\right)\hat{F}_1^*$ then $\hat{F}_1\hat{F}_2$ is soft $\hat{F}-$ quasi $\hat{K}-$normal operator.

Proof: $(\widehat{F_1}\widehat{F_2})^*(\left((\widehat{F_1}\widehat{F_2})^*\right)^v(\widehat{F_1}\widehat{F_2}))=(\widehat{F_2}\widehat{F_1})^*\left((\widehat{F_1}\widehat{F_2})^*\right)^v(\widehat{F_1}\widehat{F_2})$

$$= (\hat{F}_1^* \hat{F}_2^*)\left((\hat{F}_2 \hat{F}_1)^*\right)^v (\hat{F}_1 \hat{F}_2) = \hat{F}_1^* (\hat{F}_2^* \hat{F}_1^{*v})\hat{F}_2^{*v} (\hat{F}_1 \hat{F}_2)$$
$$= \hat{F}_1^* \left(\hat{F}_1^v \hat{F}_2\right)^* \hat{F}_2^{*v} \left(\hat{F}_1 \hat{F}_2\right) = \hat{F}_1^* \hat{F}_1^{*v} \hat{F}_2^* (\hat{F}_2^{*v} \hat{F}_1) \hat{F}_2$$
$$= \hat{F}_1^* \hat{F}_1^{*v} (\hat{F}_2^* \hat{F}_1) \hat{F}_2^{*v} \hat{F}_2 = \hat{F}_1^* \hat{F}_1^{*v} \hat{F}_1 \hat{F}_2^* \hat{F}_2^{*v} \hat{F}_2$$
$$= \hat{F}_1^* (\hat{F}_1^{*v} \hat{F}_1) \hat{F}_2^* (\hat{F}_2^{*v} \hat{F}_2) = \hat{K}_1 (\hat{F}_1^{*v} \hat{F}_1) \hat{F}_1^* \hat{K}_2 (\hat{F}_2^{*v} \hat{F}_2) \hat{F}_2^*$$
$$= \hat{K}_1 \hat{K}_2 (\hat{F}_1^{*v} \hat{F}_1) \hat{F}_1^* (\hat{F}_2^{*v} \hat{F}_2) \hat{F}_2^*, \text{ Let } \hat{K} = \hat{K}_1 \hat{K}_2$$
$$= \hat{K} \hat{F}_1^{*v} \hat{F}_1 \left(\hat{F}_2^v \hat{F}_1\right)^* \hat{F}_2 \hat{F}_2^* = \hat{K} \hat{F}_1^{*v} \hat{F}_1 \left(\hat{F}_1 \hat{F}_2^v\right)^* \hat{F}_2 \hat{F}_2^*$$
$$= \hat{K} \hat{F}_1^{*v} \hat{F}_1 \hat{F}_2^{*v} (\hat{F}_1^* \hat{F}_2) \hat{F}_2^* = \hat{K} \hat{F}_1^{*v} \hat{F}_2^{*v} \hat{F}_1 \hat{F}_1^* \hat{F}_2 \hat{F}_2^*$$
$$= \hat{K} \hat{F}_1^{*v} \hat{F}_2^{*v} \hat{F}_1 \hat{F}_2 \hat{F}_1^* \hat{F}_2^* = \hat{K} \left(\hat{F}_1 \hat{F}_2\right)^{*v} (\hat{F}_1 \hat{F}_2) \left(\hat{F}_1 \hat{F}_2\right)^*$$

Therefore, the product $\hat{F}_1 \hat{F}_2$ is soft $\hat{F}-$ quasi normal operator.

Theorem 3.11: *Let* $\hat{F}_1, \hat{F}_2, \dots, \hat{F}_v : SE(H) \to SE(H)$ *was soft* $\hat{F}-$*quasi* $\hat{K}-$*normal operators then* $\hat{F}_1 \oplus \hat{F}_2 \oplus \dots \oplus \hat{F}_v$ *is soft* $\hat{F}-$ *quasi* $\hat{K}-$*normal operator.*

Proof:

$$\left(\hat{F}_1 \oplus \hat{F}_2 \oplus \dots \oplus \hat{F}_v\right)^* \left(\left(\left(\widehat{F_1} \oplus \widehat{F_1} \oplus \dots \oplus \widehat{F_v}\right)^*\right)^v \left(\widehat{F_1} \oplus \widehat{F_2} \oplus \dots \oplus \widehat{F_v}\right)\right)$$
$$= \left(\hat{F}_1^* \oplus \hat{F}_2^* \oplus \dots \oplus \hat{F}_v^*\right)\left(\left(\hat{F}_1^*\right)^v \oplus \left(\hat{F}_2^*\right)^v \oplus \dots \oplus \left(\hat{F}_v^*\right)^v\right) (\hat{F}_1 \oplus \hat{F}_2 \oplus \dots \oplus \hat{F}_v)$$
$$= \left(\hat{F}_1^* \left(\hat{F}_1^*\right)^v \hat{F}_1 \oplus \hat{F}_2^* \left(\hat{F}_2^*\right)^v \hat{F}_2 \oplus \dots \oplus \hat{F}_v^* \left(\hat{F}_v^*\right)^v \hat{F}_v\right)$$
$$= \left(\hat{\kappa}_1 \left(\hat{F}_1^*\right)^v \hat{F}_1 \hat{F}_1^* \oplus \hat{\kappa}_2^* \left(\hat{F}_2^*\right)^v \hat{F}_2 \hat{F}_2^* \oplus \dots \oplus \hat{\kappa}_v^* \left(\hat{F}_v^*\right)^v \hat{F}_v \hat{F}_v^*\right)$$
$$= (\hat{\kappa}_1 \oplus \hat{\kappa}_1 \oplus \dots \oplus \hat{\kappa}_1)\left(\left(\hat{F}_1^*\right)^v \oplus \left(\hat{F}_2^*\right)^v \oplus \dots \oplus \left(\hat{F}_v^*\right)^v\right)$$
$$(\hat{F}_1 \oplus \hat{F}_2 \oplus \dots \oplus \hat{F}_v)\left(\hat{F}_1^* \oplus \hat{F}_2^* \oplus \dots \oplus \hat{F}_v^*\right),$$

Let $\hat{K} = (\hat{\kappa}_1 \oplus \hat{\kappa}_1 \oplus \dots \oplus \hat{\kappa}_1)$

$$= \hat{K}\left(\left(\hat{F}_1 \oplus \hat{F}_2 \oplus \dots \oplus \hat{F}_v\right)^*\right)^v \left(\hat{F}_1 \oplus \hat{F}_2 \oplus \dots \oplus \hat{F}_v\right)$$
$$\left(\hat{F}_1 \oplus \hat{F}_2 \oplus \dots \oplus \hat{F}_v\right)^*$$

So, one can obtain $\hat{F}_1 \oplus \hat{F}_2 \oplus \dots \oplus \hat{F}_v$ is soft $\hat{F}-$ quasi $\hat{K}-$normal operator.

Theorem 3.12: *Let* $\hat{F}_1, \hat{F}_2, \dots, \hat{F}_v : SE(H) \to SE(H)$ *was soft* $\hat{F}-$*quasi* $\hat{K}-$*normal operators then* $\hat{F}_1 \otimes \hat{F}_2 \otimes \dots \otimes \hat{F}_v$ *is soft* $\hat{F}-$*quasi* $\hat{K}-$*normal operator.*

Proof:

$$\left(\widehat{F_1} \otimes \widehat{F_2} \otimes \dots \otimes \widehat{F_v}\right)^* \left(\left(\left(\widehat{F_1} \otimes \widehat{F_2} \otimes \dots \otimes \widehat{F_v}\right)^*\right)^v \left(\widehat{F_1} \otimes \widehat{F_2} \otimes \dots \otimes \widehat{F_v}\right)\right)$$

$$= \left(\hat{F}_1^* \otimes \hat{F}_2^* \otimes \dots \otimes \hat{F}_v^*\right)\left(\left(\hat{F}_1^*\right)^v \otimes \left(\hat{F}_2^*\right)^v \otimes \dots \otimes \left(\hat{F}_v^*\right)^v\right)\left(\hat{F}_1 \otimes \hat{F}_2 \otimes \dots \otimes \widehat{F_v}\right)$$
$$= \left(\hat{F}_1^* \left(\hat{F}_1^*\right)^v \hat{F}_1 \otimes \hat{F}_2^* \left(\hat{F}_2^*\right)^v \hat{F}_2 \otimes \dots \otimes \hat{F}_v^* \left(\hat{F}_v^*\right)^v \hat{F}_v\right)$$
$$= \hat{K}\left(\left(\hat{F}_1^*\right)^v \otimes \left(\hat{F}_2^*\right)^v \otimes \dots \otimes \left(\hat{F}_v^*\right)^v\right)\left(\widehat{F_1} \otimes \widehat{F_2} \otimes \dots \otimes \widehat{F_v}\right)\left(\hat{F}_1^* \otimes \hat{F}_2^* \otimes \dots \otimes \hat{F}_v^*\right)$$
$$= \hat{K}\left(\left(\left(\hat{F}_1 \otimes \hat{F}_2 \otimes \dots \otimes \hat{F}_v\right)^*\right)^v \left(\hat{F}_1 \otimes \hat{F}_2 \otimes \dots \otimes \hat{F}_v\right)\left(\hat{F}_1 \otimes \hat{F}_2 \otimes \dots \otimes \hat{F}_v\right)^*\right)$$

So, one can obtain $\hat{F}_1 \otimes \hat{F}_2 \otimes \dots \otimes \hat{F}_v$ is soft $\hat{F}-$quasi $\hat{K}-$normal operator.

4. Discussion and Conclusion

The importance of the operator in functional analysis in terms of algebra and analysis has been linked to soft set theory and many spaces, one of which is the soft Hilbert space and the main coincidence in this work is the introduction of another and different set of soft Hilbert space operators, which is a Novel Type of Extended Soft quasinormal for Normal Operator in Hilbert Space Studying some of the characteristics and theories that are more effective, clear and comprehensive, and this system is expanded through other studies and research that are deeper in this direction.

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Received December, 2024