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Mathematical modelling of the spread of Nipah virus in bats, humans and pigs

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Abstract

A mathematical model formulated using ordinary differential equations is proposed to study the dynamics of zoonotic Nipah virus in fruit bats (natural hosts of Nipah virus), pigs and humans. The presented model has one disease free equilibrium (DFE) and three endemic equilibria (EE). The next generation matrix method is employed to compute the basic reproduction number of the model $\mathcal{R}_0$. The calculations give three partial basic reproduction numbers, maximum of the three partial basic reproduction numbers is taken as the basic reproduction number of the model. The global stability of the DFE when $\mathcal{R}_0 \leq 1$ is proved using a suitable Lyapunov function. The graph theoretic method is utilized to show the global stability of the three EE under certain conditions on the parameters that affect the basic reproduction numbers.

Subject Classification (2020): *97M10.*

Keywords: *Mathematical modelling, Ordinary differential equations, Basic reproduction number, Lyapunov functions, Graph theoretic method, Matrix theoretic method, Nipah virus, Zoonotic diseases, Next generation matrix method, Stability of equilibria.*

1. Introduction

Nipah virus (Niv) is a highly virulent zoonotic virus. It was initially discovered in September 1998 during an upsurge of acute feverish illness

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with encephalitis among pig farmers in Malaysia [3, 6, 23, 30]. Simultaneously, local pigs were distressed with conspicuous respiratory and nervous disease [6, 9]. Subsequently, other outbreaks of Niv have been reported from India, Bangladesh and Singapore. In 2001, 66 human cases of encephalitis occurred in India, with around 74% fatal cases [6]. Another outbreak in India occurred in 2007, with five cases which were all fatal [6]. On 19 May 2018, an outbreak of Niv in India was reported to WHO, as of 1 June 2018, there were 18 confirmed cases and 17 deaths. Bangladesh has experienced outbreaks nearly every year between 2001 and 2014 [6].

Nipah virus is considered to be non-pathogenic in its natural hosts (fruit bats) [6]. The most predominant route is indirect. Pig farmers at times plant fruit trees in close proximity to pig sties in order to provide shade for pigs. The bats are attracted by fruits and leave behind samples of Niv as they chew on fruits [8]. The infection is then passed to pigs and pig farmers. Other pigs can get infection from infected pigs, humans may get infected after contact with infected pigs, bats, or humans [1, 30]. Moreover, ingesting fruits or fruit products mixed with urine or saliva from infected fruit bats is another route of infection for humans [9]. In humans, after exposure and an incubation period of 4 to 14 days, illness manifests with 3-14 days of fever, cough, and headache, accompanied by doziness and mental confusion. These signs and symptoms can progress to coma within 24-48 hours, and ultimately death in severe case. Some patients experienced respiratory illness during the early part of the infection, and half of the patients showing serious neurological signs showed also pulmonary signs [6, 9, 15, 23, 30]. Time from onset of symptoms to death in fatal cases was approximately 16 days [6]. Most people who recover from Niv make a full recovery, however, a small portion of patients who recover subsequently relapse. The virus is highly contagious in pigs. Pigs can spread the virus without exhibiting symptoms. Niv disease manifestation in pigs include neurological and respiratory diseases, following an incubation of 4 to 14 days [6, 9].

There is no certified vaccine or treatment for human use. Prevention strategies include early implementation of effective quarantines and control of animal movements, discouraging planting of fruit trees on or in the neighbourhood of pig farms, boiling freshly collected date palm juice, and properly washing and peeling fruits before eating [6].

Mathematical modelling has become a valuable tool in the analysis of dynamics of infectious diseases, providing useful insights concerning transmission patterns and detection of parameters to mitigate infectious diseases. Well-parametrized mathematical models enable us to guess the

state and advancement of an outbreak from known information, experiment on sorts of viable control plans and their effectiveness virtually before implementing.

Various studies on pathology and epidemiology of Niv have been conducted, several mathematical models have been presented as follows; in [31], authors proposed an SIRD model to examine the aftermath of exposed contact with dead bodies of infected individuals prior to burial or cremation and their disposal rate on the dynamics of Nipah virus infection. The most favourable use of control plans to ease the spread of Nipah virus using optimal control technique is studied in [27]. SEI model with interactions between bat and human populations is presented in [22] to research the communication of Nipah virus. In [17], authors presented SEITR model to study the transmission of Nipah virus in human population. In [5], authors studied a pig-human Nipah model to understand the disease spillover from pigs to humans. In [25], SEIR and SEIRV models are presented to analyse the effect of a vaccine in prevention and recovery from Nipah virus in humans and animals. To study the dynamics of Niv, compartmental model considering bat and human populations is presented in [21]. Associated studies include the susceptible-infected type mathematical model for HIV/AIDS presented in [18], and Nigeria case study of mathematical modelling of COVID-19 pandemic disease with some non-pharmaceutical interventions presented in [16]. Other closely related studies are presented in [4, 13, 19, 28].

There are several Nipah models in the literature, some consider the transmission of Niv in human population only, very few consider transmission in bat and human population. Most of these models, if not all, only consider the direct transmission route of Niv. As mentioned in the literature, the most predominant route for the transmission of Nipah virus is indirect, pigs and humans get the infection mostly from consuming contaminated fruits. We wish to enhance the existing literature by proposing a deterministic model that considers three population species, the fruit bats (natural hosts of Niv), the pigs, and the humans. Most importantly, our model will take into account the indirect transmission route in both pigs and humans. We intend to present rigorous mathematical analysis of the model that will give insightful understanding of the conditions under which the infection will persist in any of the population species considered.

Henceforth, the structure of the paper is as follows: In section 2, we put together the mathematical model and find the steady states. In section 3, we compute the basic reproduction number and present the stability

analysis of the steady states. In section 4, we present the numerical simulation of the results, and section 5 is discussion and concluding remarks.

2. Mathematical modelling

We employ ordinary differential equations to formulate the mathematical model for the transmission of Nipah virus.

2.1 *Model formulation*

The model consists of eight compartments; susceptible bats, pigs and humans (S_b, S_p and S_h, respectively), exposed humans (E_h), infectious bats, pigs and humans (I_b, I_p and I_h, respectively), and the B compartment representing the concentration of virus on the environment. We do not have exposed class in pigs and bats because pigs are infectious even without showing any symptoms, and bats are natural hosts and do not exhibit any signs of infection during the infectious state. The total populations of bats, pigs and humans are $N_b(t) = S_b(t) + I_b(t)$, $N_p(t) = S_p(t) + I_p(t)$, and $N_h(t) = S_h(t) + E_h(t) + I_h(t)$ respectively.

The bat natural death rate is μ_b and the bat recruitment rate is Λ_b. The effective contact rate of susceptible bats with infectious bats is β_{bb}.

The rates at which infectious bats shed virus on the environment is ε_b. The rate at which virus is eliminated from the environment through sanitation, loss of virulence or sterilisation is δ.

The pig death rate (death due to reason not related to the infection) is μ_p and the pig recruitment rate is Λ_p. The rate at which pigs die due to the disease is Γ_p, the rate at which susceptible pigs become infectious through direct contact with infectious pigs is β_{pp}, and the rate at which susceptible pigs become infectious through ingesting virus from the environment is τ_p.

The human natural death rate is μ_h and the human recruitment rate is Λ_h. The rate at which humans die due to the disease is Γ_h, the rates at which susceptible humans become exposed due to direct contact with infectious pigs and humans are β_{ph} and β_{hh}, respectively. The rate at which susceptible humans become exposed through ingesting virus from the environment is τ_h, and the rate at which exposed humans become infectious is γ.

Following the above discussion, our mathematical model describing the transmission dynamics of Nipah virus in bats, pigs and humans population is presented as

$$\frac{dS_b}{dt} = \Lambda_b - \beta_{bb}\frac{S_b I_b}{N_b} - \mu_b S_b, \tag{2.1}$$

$$\frac{dI_b}{dt} = \frac{\beta_{bb} S_b I_b}{N_b} - \mu_b I_b, \tag{2.2}$$

$$\frac{dS_p}{dt} = \Lambda_p - \beta_{pp}\frac{S_p I_p}{N_p} - \tau_p \frac{S_p B}{\kappa + B} - \mu_p S_p, \tag{2.3}$$

$$\frac{dI_p}{dt} = \beta_{pp}\frac{S_p I_p}{N_p} + \tau_p \frac{S_p B}{\kappa + B} - (\Gamma_p + \mu_p) I_p, \tag{2.4}$$

$$\frac{dS_h}{dt} = \Lambda_h - \beta_{ph}\frac{I_p S_h}{N_p} - \beta_{hh}\frac{S_h I_h}{N_h} - \tau_h \frac{B S_h}{B + \kappa} - \mu_h S_h, \tag{2.5}$$

$$\frac{dE_h}{dt} = \beta_{ph}\frac{I_p S_h}{N_p} + \beta_{hh}\frac{S_h I_h}{N_h} + \tau_h \frac{B S_h}{B + \kappa} - (\gamma + \mu_h) E_h, \tag{2.6}$$

$$\frac{dI_h}{dt} = \gamma E_h - (\mu_h + \Gamma_h) I_h, \tag{2.7}$$

$$\frac{dB}{dt} = \varepsilon_b I_b - \delta B, \tag{2.8}$$

with non-negative initial condition

$$\begin{aligned}&(S_b(0), I_b(0), S_p(0), I_p(0), S_h(0), E_h(0), I_h(0), B(0))\\ &= (S_{b0}, I_{b0}, S_{p0}, I_{p0}, S_{h0}, E_{h0}, I_{h0}, B_0).\end{aligned} \tag{2.9}$$

The domain of our model is defined as the invariant set

$$\begin{aligned}\Omega = \Bigg\{&(S_b, I_b, S_p, I_p, S_h, E_h, I_h, B) \in \mathbb{R}_+^8 \mid N_b(t) \le \frac{2\Lambda_b}{\mu_b}, N_p(t) \le \\ &\frac{\Lambda_p}{\mu_p} + \frac{\Lambda_p}{\Gamma_p + \mu_p}, N_h(t) \le \frac{\Lambda_h}{\mu_h} + \frac{\Lambda_h}{\gamma + \mu_h} + \frac{\gamma \Lambda_h}{(\mu_h + \Gamma_h)(\gamma + \mu_h)} \text{ and } B \le \frac{\varepsilon_b \Lambda_b}{\delta \mu_b}\Bigg\}.\end{aligned}$$

2.2 *Well-posedness of the model*

We show that our model can be used to deal with real life outbreaks.

Theorem 2.1: *System (2.1)-(2.8), with non-negative initial conditions (2.9) has a unique solution that is non-negative and bounded in* Ω.

Proof: **Existence and uniqueness of solution:** The right sides of equations (2.1)-(2.8) are continuously differentiable functions of the respective state variables, thus they are locally Lipschitz continuous. Hence there exists

a unique solution for system (2.1)-(2.8) defined in some time interval containing the initial time $t = 0$.

Non-negativity of solution: We note that we start with non-negative initial conditions. Let t_0 be the smallest time for which any of the state variables is zero. When $S_b(t_0) = 0$, from (2.1), we get $S_b' = \Lambda_b > 0$. This means that $S_b(t)$ is an increasing function in some interval containing t_0 hence $S_b(t) > 0$ in the neighbourhood of t_0.

When $I_b(t_0) = 0$, from (2.2), $I_b' = 0$. This means that in the neighbourhood of t_0, $I_b(t)$, stays at zero, thus it is never negative. From (2.8), when $B(t_0) = 0$, $B' = \varepsilon_b I_b \geq 0$, so in the neighbourhood of t_0, $B(t)$ is zero. From (2.3), when $S_p(t_0) = 0$, $S_p' = \Lambda_p > 0$, thus in the neighbourhood of t_0, $S_p(t) > 0$. In the same manner, it can be shown from (2.4), (2.5), (2.6) and (2.7) that if $I_p(t_0) = 0$, $S_h(t_0) = 0$, $E_h(t_0) = 0$ or $I_h(t_0) = 0$, then $I_p' \geq 0$, $S_h' > 0$, $E_h' > 0$ or $I_h' > 0$, thus in the neighbourhood of t_0, I_p, S_h, E_h and I_h are non negative functions.

Boundedness of solution: At equilibrium, equation (2.1) can be written as $\Lambda_b - \mu_b S_b = \beta_{bb} \frac{S_b I_b}{N_b}$. Since S_b and I_b are non-negative, it follows that $\Lambda_b - \mu_b S_b \geq 0$ $\frac{\Lambda_b}{\mu_b} \geq S_b$. Adding equations (2.1) and (2.2), we also get $I_b \leq \frac{\Lambda_b}{\mu_b}$. At equilibrium, equation (2.3) can be rearranged as $\Lambda_p - \mu_p S_p = \beta_{pp} \frac{S_p I_p}{N_p} + \tau_p \frac{S_p B}{\kappa + B} \geq 0$, thus $S_p \leq \frac{\Lambda_p}{\mu_p}$. Adding equations (2.3) and (2.4) at equilibrium, it can be shown that $I_p \leq \frac{\Lambda_p}{\mu_p}$. Similarly from (2.5), we get $S_h \leq \frac{\Lambda_h}{\mu_h}$, and adding (2.5) and (2.7), we get $I_h \leq \frac{\Lambda_h}{\mu_h}$. From equation (2.6) and (2.8) respectively, we get $E_h \leq \frac{(\mu_h + \Gamma_h)\Lambda_h}{\gamma \mu_h}$ and $B \leq \frac{\varepsilon_b \Lambda_b}{\delta \mu_b}$

2.3 *Finding the disease free equilibrium*

To find the steady states, we equate equations (2.1)-(2.8) to zero and solve for the state variables. From (2.2), we get

$$S_b = \frac{N_{b0}\mu_b}{\beta_{bb}} \tag{2.10}$$

or

$$I_b = 0.$$

When $I_b = 0$, from (2.1), $S_b = \frac{\Lambda_b}{\mu_b}$ and from (2.8), we get $B = 0$. Substituting $B = 0$ into (2.4), we get $I_p = 0$ or

$$S_p = \frac{(\mu_p + \Gamma_p)N_{p0}}{\beta_{pp}} \tag{2.11}$$

When $I_p = 0$, from (2.3), we get $S_p = \frac{\Lambda_p}{\mu_p}$. Solving for E_h and substituting into (2.6), we get $I_h = 0$ or

$$S_h = \frac{(\mu_h+\gamma)(\mu_h+\Gamma_h)N_{h0}}{\gamma\beta_{hh}} \tag{2.12}$$

When $I_h = 0$, from (2.5), we get $S_h = \frac{\Lambda_h}{\mu_h}$. Therefore the DFE is $X_0 = (S_b, I_b, S_p, I_p, S_h, E_h, I_h, B) = \left(\frac{\Lambda_b}{\mu_b}, 0, \frac{\Lambda_p}{\mu_p}, 0, \frac{\Lambda_h}{\mu_h}, 0, 0, 0\right)$.

We leave equations (2.10), (2.11) and (2.12) for further exploration at a later stage, for now, we proceed to calculate the basic reproduction number of the model.

3. Stability analysis

Theorem 3.1: *The basic reproduction number of the model* $\mathcal{R}_0 = \rho(FV^{-1}) = \max\{R_1, R_2, R_3\}$ *where*

$$R_1 = \frac{\beta_{bb}}{\mu_b} \tag{3.1}$$

represents the basic reproduction number in bat population,

$$R_2 = \frac{\beta_{pp}}{\mu_p+\Gamma_p} \tag{3.2}$$

represents the basic reproduction number in pig population, and

$$R_3 = \frac{\gamma\beta_{hh}}{(\mu_h+\Gamma_h)(\gamma+\mu_h)} \tag{3.3}$$

represents the basic reproduction number in human population.

Proof: See Appendix A

3.1 *Stability of DFE*

3.1.1 Local stability of DFE

Theorem 3.2: *The DFE* $E_0 = \left(\frac{\Lambda_b}{\mu_b}, 0, \frac{\Lambda_p}{\mu_p}, 0, \frac{\Lambda_h}{\mu_h}, 0, 0, 0\right)$ *is locally asymptotically stable if* $\mathcal{R}_0 \leq 1$, *and unstable if* $\mathcal{R}_0 > 1$.

Proof: See proof of Theorem 2 of [29]

3.1.2 Global stability of DFE

Theorem 3.3: *When* $\mathcal{R}_0 \leq 1$, *the disease free equilibrium is globally asymptotically stable in* Ω.

Proof: We use matrix-theoretic method to prove global stability of the DFE, see [24] for more details. Multiplying matrix F with the inverse of matrix V as given in (A.1) and (A.2) respectively, and then substituting (A.3), (A.4) and (A.5), we get

$$V^{-1}F = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & \frac{\tau_p \Lambda_p}{\mu_p \kappa(\mu_p+\gamma_p)} \\ 0 & \frac{\beta_{ph}\Lambda_h \mu_p}{\mu_h \Lambda_p(\gamma+\mu_h)} & 0 & \frac{\beta_{hh}}{\gamma+\mu_h} & \frac{\tau_h \lambda_h}{\mu_h \kappa(\gamma+\mu_h)} \\ 0 & \frac{\gamma\beta_{ph}\Lambda_h \mu_p}{\mu_h \Lambda_p(\gamma+\mu_h)(\mu_h+\Gamma_h)} & 0 & R_3 & \frac{\gamma\tau_h \Lambda_h}{\mu_h \kappa(\gamma+\mu_h)(\mu_h+\Gamma_h)} \\ \frac{\varepsilon_b R_1}{\delta} & 0 & 0 & 0 & 0 \end{bmatrix}$$

We note that the digraph of $V^{-1}F$ is not strongly connected, thus $V^{-1}F$ is reducible.

Let $W^T = (w_1, w_2, w_3, w_4, w_5)$ be the left eigenvector of $V^{-1}F$ corresponding to $\mathcal{R}_0$, then

$$(w_1, w_2, w_3, w_4, w_5)V^{-1}F = \mathcal{R}_0(w_1, w_2, w_3, w_4, w_5)$$

$$\left(w_1 R_1 + \frac{\varepsilon_b R_1 w_5}{\delta}, w_2 R_2 + \frac{\beta_{ph}\Lambda_h \mu_p w_3}{\mu_h \Lambda_p(\gamma+\mu_h)} + \frac{\gamma\beta_{ph}\Lambda_h \mu_p w_4}{\mu_h \Lambda_p(\mu_h+\Gamma_h)(\gamma+\mu_h)}, 0, \frac{\beta_{hh} w_3}{\gamma+\mu_h} + w_4 R_3, \right.$$

$$\left. \frac{\tau_p \Lambda_p w_2}{\mu_p(\mu_p+\Gamma_p)\kappa} + \frac{\tau_h \Lambda_h w_3}{\mu_h \kappa(\gamma+\mu_h)} + \frac{\gamma\tau_h N_{h0} w_4}{\kappa(\mu_h+\Gamma_h)(\gamma+\mu_h)} \right) = \mathcal{R}_0(w_1, w_2, w_3, w_4, w_5)$$

This means that we have the following system of equations:

$$w_1 R_1 + \frac{\varepsilon_b R_1 w_5}{\delta} = \mathcal{R}_0 w_1 \tag{3.4}$$

$$w_2 R_2 + \frac{\beta_{ph}\Lambda_h \mu_p w_3}{\mu_h \Lambda_p(\gamma+\mu_h)} + \frac{\gamma\beta_{ph}\Lambda_h \mu_p w_4}{\mu_h \Lambda_p(\mu_h+\Gamma_h)(\gamma+\mu_h)} = \mathcal{R}_0 w_2 \tag{3.5}$$

$$0 = \mathcal{R}_0 w_3 \tag{3.6}$$

$$\frac{\beta_{hh} w_3}{\gamma+\mu_h} + w_4 R_3 = \mathcal{R}_0 w_4 \tag{3.7}$$

$$\frac{\tau_p \Lambda_p w_2}{\mu_p(\mu_p+\Gamma_p)\kappa} + \frac{\tau_h \Lambda_h w_3}{\mu_h \kappa(\gamma+\mu_h)} + \frac{\gamma\tau_h \Lambda_h w_4}{\mu_h \kappa(\mu_h+\Gamma_h)(\gamma+\mu_h)} = \mathcal{R}_0 w_5 \tag{3.8}$$

From (3.6), $w_3 = 0$. Substituting w_3 into equations (3.5), (3.7) and (3.8), we get

$$w_2 R_2 + \frac{\gamma \beta_{ph} \Lambda_h \mu_p w_4}{\mu_h \Lambda_p (\mu_h + \Gamma_h)(\gamma + \mu_h)} = \mathcal{R}_0 w_2 \tag{3.9}$$

$$w_4 R_3 = \mathcal{R}_0 w_4 \tag{3.10}$$

$$\frac{\tau_p \Lambda_p w_2}{\mu_p (\mu_p + \Gamma_p)\kappa} + \frac{\gamma \tau_h \Lambda_h w_4}{\mu_h \kappa (\mu_h + \Gamma_h)(\gamma + \mu_h)} = \mathcal{R}_0 w_5 \tag{3.11}$$

From (3.10), either $w_4 = 0$ or $\mathcal{R}_0 = R_3$

Case 1: When $w_4 = 0$

From (3.9), we get

$w_2(R_2 - \mathcal{R}_0) = 0$, which means that either $w_2 = 0$ or $\mathcal{R}_0 = R_2$.

Case 1.1: When $w_2 = 0$

From (3.11), we get $w_5 = 0$. Then from (3.4), we get $w_1(R_1 - \mathcal{R}_0) = 0$, which means that either $w_1 = 0$ or $\mathcal{R}_0 = R_1$.

Case 1.1.1: When $w_1 = 0$

We have the trivial eigenvector $W_0^T = (0,0,0,0,0)$.

Case 1.1.2: When $\mathcal{R}_0 = R_1$

From (3.11), $w_5 = 0$. Then from (3.4), we get $w_1(R_1 - \mathcal{R}_0) = 0 w_1 \cdot 0 = 0$. So w_1 can be taken as any number, for simplicity, we take $w_1 = 1$. We note that taking $w_1 = 0$ gives the trivial solution. Therefore in this case, we have the left eigenvector as $W_1^T = (1,0,0,0,0)$.

Case 1.2: When $\mathcal{R}_0 = R_2$

From (3.9), we have $w_2 \cdot 0 = 0$. Choosing $w_2 = 1$, from (3.11), we get

$$w_5 = \frac{\tau_p \Lambda_p}{\mu_p R_2 \kappa (\mu_p + \Gamma_p)},$$

and from (3.4), we get

$$w_1 = \frac{\varepsilon_b R_1 \tau_p \Lambda_p}{\mu_p \delta R_2 \kappa (\mu_p + \Gamma_p)(R_2 - R_1)}.$$

In this case, we have the left eigenvector as

$$W_2^T = \left(\frac{\varepsilon_b R_1 \tau_p \Lambda_p}{\mu_p \delta R_2 \kappa (\mu_p + \Gamma_p)(R_2 - R_1)}, 1, 0, 0, \frac{\tau_p \Lambda_p}{\mu_p R_2 \kappa (\mu_p + \Gamma_p)} \right)$$

Case 2: When $\mathcal{R}_0 = R_3$

From (3.10), we get $w_4 \cdot 0 = 0$. Taking $w_4 = 1$, from (3.9), we get

$$w_2 = \frac{\gamma\beta_{ph}\Lambda_h\mu_p}{\mu_h\Lambda_p(\mu_h+\Gamma_h)(\gamma+\mu_h)(R_3-R_2)}.$$

From (3.11), we get

$$w_5 = \frac{1}{R_3}\left[\frac{\tau_p\gamma\beta_{ph}\Lambda_h}{\mu_h(\mu_p+\Gamma_p)\kappa(\mu_h+\Gamma_h)(\gamma+\mu_h)(R_3-R_2)} + \frac{\gamma\tau_h\Lambda_h}{\mu_h\kappa(\mu_h+\Gamma_h)(\gamma+\mu_h)}\right],$$

then from (3.4), we get

$$w_1 = \frac{\varepsilon_b R_1}{\delta R_3(R_3-R_1)}\left[\frac{\tau_p\gamma\beta_{ph}\Lambda_h}{\mu_h(\mu_p+\Gamma_p)\kappa(\mu_h+\Gamma_h)(\gamma+\mu_h)(R_3-R_2)} + \frac{\gamma\tau_h\Lambda_h}{\mu_h\kappa(\mu_h+\Gamma_h)(\gamma+\mu_h)}\right].$$

Thus in this case, we have the left eigenvalue as

$$W_3^T = \left(\frac{\varepsilon_b R_1}{\delta(R_3-R_1)}\mathcal{A}, \frac{\gamma\beta_{ph}\Lambda_h\mu_p}{\mu_h\Lambda_p(\mu_h+\Gamma_h)(\gamma+\mu_h)(R_3-R_2)}, 0, 1, \mathcal{A}\right)$$

where

$$\mathcal{A} = \frac{1}{R_3}\left[\frac{\tau_p\gamma\beta_{ph}\Lambda_h}{\mu_h(\mu_p+\Gamma_p)\kappa(\mu_h+\Gamma_h)(\gamma+\mu_h)(R_3-R_2)} + \frac{\gamma\tau_h\Lambda_h}{\mu_h\kappa(\mu_h+\Gamma_h)(\gamma+\mu_h)}\right]$$

We note that for all the non-trivial eigenvectors, all the components are non-negative, and at least one of the components is positive. The left eigenvector W^T can be considered as W_1^T, W_2^T or W_3^T.

We now consider the function

$$\begin{aligned} Q &= W^T V^{-1} x \\ &= \left(\frac{w_1}{\mu_b} + \frac{\varepsilon_b w_5}{\mu_b\delta}\right) I_b + \frac{w_2}{\mu_p+\Gamma_p} I_p + \frac{\gamma w_4}{(\gamma+\mu_h)(\mu_h+\Gamma_h)} E_h + \frac{w_4}{\mu_h+\Gamma_h} I_h + \frac{w_5}{\delta} B. \end{aligned}$$

Whose derivative along the trajectories of the system is

$$Q' = (\mathcal{R}_0 - 1)W^T x - W^T V^{-1} f, \tag{3.12}$$

where

$$f = (F - V)x - \mathcal{F} + \mathcal{V}$$
$$= \begin{bmatrix} \beta_{bb} I_b \left(1 - \frac{S_b}{N_{b0}}\right) \\ \beta_{pp} I_p \left(1 - \frac{S_p}{N_{p0}}\right) + \tau_p B \left(\frac{\Lambda_p}{\mu_p \kappa} - \frac{S_p}{\kappa + B}\right) \\ \frac{\beta_{ph} I_p}{N_{p0}} (N_{h0} - S_h) + \beta_{hh} I_h \left(1 - \frac{S_h}{N_{h0}}\right) + \tau_h B \left(\frac{\Lambda_h}{\mu_h \kappa} - \frac{S_h}{\kappa + B}\right) \\ 0 \\ 0 \end{bmatrix} \geq 0.$$

It can be shown that

$$W^T V^{-1} f$$
$$= \left(\frac{w_1}{\mu_b} + \frac{\varepsilon_b w_5}{\mu_b \delta}\right)\left(1 - \frac{S_b}{N_{b0}}\right)\beta_{bb} I_b + \left[\beta_{pp} I_p \left(1 - \frac{S_p}{N_{p0}}\right) + \tau_p B \left(\frac{\Lambda_p}{\mu_p \kappa} - \frac{S_p}{\kappa + B}\right)\right] \frac{w_2}{\mu_p + \Gamma_p} +$$
$$\left[\frac{\beta_{ph} I_p}{N_{p0}} (N_{h0} - S_h) + \beta_{hh} I_h \left(1 - \frac{S_h}{N_{h0}}\right) + \tau_h B \left(\frac{\Lambda_h}{\mu_h \kappa} - \frac{S_h}{\kappa + B}\right)\right] \frac{w_4 \gamma}{(\gamma + \mu_h)(\mu_h + \Gamma_h)} \geq 0.$$

Since $W^T V^{-1} f$ is non-negative and the first term of (3.12) is non-positive when $\mathcal{R}_0 \leq 1$, $Q' \leq 0$ when $\mathcal{R}_0 \leq 1$. Which means that Q is a non-increasing function along the trajectories of the system when $\mathcal{R}_0 \leq 1$, hence it is the Lyapunov function of the system when $\mathcal{R}_0 \leq 1$.

We recall that we have three possibilities of Q, where $W^T = W_1^T$, $W^T = W_2^T$ or $W^T = W_3^T$. However, by LaSalle's invariance principle [14], the DFE E_0 will be globally asymptotically stable if the following conditions are met:

1. $Q = 0$ for $X = E_0$ and $Q > 0$ for all $X \neq E_0$
2. $Q' < 0$ for all $X \neq E_0$ and that {X in the domain of the system $: Q' = 0\} = \{E_0\}$

Looking at the first condition, Q is obviously non-negative. For it to be zero, each term of Q must be zero. That is

$$\left(\frac{w_1}{\mu_b} + \frac{\varepsilon_b w_5}{\mu_b \delta}\right) I_b = 0 \tag{3.13}$$

$$\frac{w_2}{\mu_p + \Gamma_p} I_p = 0 \tag{3.14}$$

$$\frac{\gamma w_4}{(\gamma + \mu_h)(\mu_h + \Gamma_h)} E_h = 0 \tag{3.15}$$

$$\frac{w_4}{\mu_h+\Gamma_h}I_h = 0 \tag{3.16}$$

$$\frac{w_5}{\delta}B = 0 \tag{3.17}$$

For Q to be zero only at the DFE, From (3.13), we must have at least one of w_1 and w_5 as non-zero, so that we are sure that the factor that makes the left side of (3.13) zero is I_b. All three options W_1^T, W_2^T and W_3^T satisfy this condition. Moving on to equation (3.14), w_2 must be non-zero so that we have $I_p = 0$. This condition is only satisfied by W_2^T and W_3^T, since the second component of W_1^T is zero. So, we are now left with two options for W^T, W_2^T or W_3^T. Next we look at equations (3.15) and (3.16) and realise that w_4 must be non-zero so that we have $E_h = 0$ and $I_h = 0$. Since the fourth component of W_2^T is zero, we are only left with W_3^T as a possible choice for W^T. Lastly, from equation (3.17), w_5 must be non-zero so that we have $B = 0$. This last condition is also met by W_3^T, therefore the left eigenvector that we need is $W^T = W_3^T$. We recall that $W^T = W_3^T$ implies that $\mathcal{R}_0 = R_3$.

Now that we know that $Q = W_3^T V^{-1} x$, and that the first condition of the LaSalle's invariance principle is satisfied, we look at the second condition for the global stability of the DFE. That is, we want to show that the DFE is the only point in the domain of system where $Q' = 0$. When $Q' = 0$, we have that

$$\left(\frac{w_1}{\mu_b}+\frac{\varepsilon_b w_5}{\mu_b\delta}\right)\left(1-\frac{S_b}{N_{b0}}\right)\beta_{bb}I_b + \left[\beta_{pp}I_p\left(1-\frac{S_p}{N_{p0}}\right)+\tau_p B\left(\frac{\Lambda_p}{\mu_p\kappa}-\frac{S_p}{\kappa+B}\right)\right]\frac{w_2}{\mu_p+\Gamma_p} +$$
$$\left[\frac{\beta_{ph}I_p}{N_{p0}}(N_{h0}-S_h)+\beta_{hh}I_h\left(1-\frac{S_h}{N_{h0}}\right)+\tau_h B\left(\frac{\Lambda_h}{\mu_h\kappa}-\frac{S_h}{\kappa+B}\right)\right]\frac{w_4\gamma}{(\gamma+\mu_h)(\mu_h+\Gamma_h)}$$
$$= (\mathcal{R}_0-1)(w_1 I_b + w_2 I_p + w_4 I_h + w_5 B)$$

Since $\mathcal{R}_0 \le 1$, we have

$$\left(\frac{w_1}{\mu_b}+\frac{\varepsilon_b w_5}{\mu_b\delta}\right)\left(1-\frac{S_b}{N_{b0}}\right)\beta_{bb}I_b + \left[\beta_{pp}I_p\left(1-\frac{S_p}{N_{p0}}\right)+\tau_p B\left(\frac{\Lambda_p}{\mu_p\kappa}-\frac{S_p}{\kappa+B}\right)\right]\frac{w_2}{\mu_p+\Gamma_p} +$$
$$\left[\frac{\beta_{ph}I_p}{N_{p0}}(N_{h0}-S_h)+\beta_{hh}I_h\left(1-\frac{S_h}{N_{h0}}\right)+\tau_h B\left(\frac{\Lambda_h}{\mu_h\kappa}-\frac{S_h}{\kappa+B}\right)\right]\frac{w_4\gamma}{(\gamma+\mu_h)(\mu_h+\Gamma_h)} \le 0$$

Note that each term on the left-hand side is non-negative, thus the left-hand side can never be negative, but it can be zero provided that each term on the left-hand side is zero. Thus we have the following system of equations:

$$\left(\frac{w_1}{\mu_b}+\frac{\varepsilon_b w_5}{\mu_b \delta}\right)\left(1-\frac{S_b}{N_{b0}}\right)\beta_{bb} I_b = 0 \tag{3.18}$$

$$\beta_{pp} I_p \left(1-\frac{S_p}{N_{p0}}\right)\frac{w_2}{\mu_p+\Gamma_p} = 0 \tag{3.19}$$

$$\tau_p B\left(\frac{\Lambda_p}{\mu_p \kappa}-\frac{S_p}{\kappa+B}\right)\frac{w_2}{\mu_p+\Gamma_p} = 0 \tag{3.20}$$

$$\left[\frac{\beta_{ph} I_p}{N_{p0}}(N_{h0}-S_h)+\beta_{hh} I_h\left(1-\frac{S_h}{N_{h0}}\right)+\tau_h B\left(\frac{\Lambda_h}{\mu_h \kappa}-\frac{S_h}{\kappa+B}\right)\right]\frac{w_4 \gamma}{(\gamma+\mu_h)(\mu_h+\Gamma_h)} = 0 \tag{3.21}$$

In equation (3.18), since we are now sure that $W^T = W_3^T$, w_1 and w_5 are non-zero. Which means that either $S_b = N_{b0}$ or $I_b = 0$, which of course yield the same result, that is $S_b = N_{b0}$ and $I_b = 0$, using equations (2.1) and (2.2) and the fact that $N_{b0} = S_b + I_b$.

From equation (3.19), w_2 is non-zero, which means that $I_p = 0$ or $S_p = N_{p0}$. Using equations (2.4) and (2.3), and that $N_{p0} = S_p + I_p$, the two cases give the same result, that $S_p = N_{p0}$ and $I_p = 0$.

From (3.20), after substituting for S_p, we get that either $B = 0$ or $\left(\frac{\Lambda_p}{\mu_p \kappa}-\frac{N_{p0}}{\kappa+B}\right) = 0$. From both cases, we get that $B = 0$.

From (3.21), after substituting $I_p = 0$ and $B = 0$, and recalling that w_4 is non-zero, we get that either $I_h = 0$ or $S_h = N_{h0}$. In either case, using equations (2.5), (2.6), (2.7) and that $N_{h0} = S_h + E_h + I_h$, we get that $I_h = E_h = 0$ and $S_h = N_{h0}$. Meaning that the DFE is the only stationary point in the domain of the system, such that $Q' = 0$. Therefore, by LaSalle's invariance principle, the DFE is globally asymptotically stable when $\mathcal{R}_0 \leq 1$.

3.2 *Stability of Endemic equilibria*

We now go back to exploring equations (2.10), (2.11) and (2.12) to find our endemic equilibria and study their stability.

From equation (2.12)

$$S_h = \frac{(\mu_h+\gamma)(\mu_h+\Gamma_h)N_{h0}}{\gamma \beta_{hh}} = \frac{N_{h0}}{R_3}$$

Substituting S_h, B and I_p into (2.5) we get

$$I_{h1}^* = \frac{\Lambda_h}{\mu_h R_3}\left(\frac{\Lambda_h}{\mu_h}-\frac{N_{h0}}{R_3}\right) > 0 \quad \text{since} \quad \frac{\Lambda_h}{\mu_h} > \frac{N_{h0}}{R_3} = S_h.$$

Therefore, we have an endemic equilibrium given by

$$EE_1 = \left(\frac{\Lambda_b}{\mu_b}, 0, \frac{\Lambda_p}{\mu_p}, 0, \frac{N_{h0}}{R_3}, \frac{(\mu_h+\Gamma_h)I_{h1}^*}{\gamma}, I_{h1}^*, 0 \right) \tag{3.22}$$

From equation (2.11), $S_p = \frac{(\mu_p+\Gamma_p)N_{p0}}{\beta_{pp}} = \frac{N_{p0}}{R_2}$. Substituting S_p and B into (2.3), we get

$$I_{p2}^* = \frac{\Lambda_p}{R_2\mu_p}\left(\frac{\Lambda_p}{\mu_p} - \frac{N_{p0}}{R_2}\right) > 0 \quad \text{since} \quad \frac{\Lambda_p}{\mu_p} > \frac{N_{p0}}{R_2} = S_p$$

Substituting I_p, B and E_h into (2.6), we get $S_h = \frac{N_{h0}N_{p0}(\mu_h+\gamma)(\mu_h+\Gamma)I_h}{\gamma(N_{p0}\beta_{hh}I_h+\beta_{ph}N_{h0}I_{p2}^*)}$. Substituting S_h, B and I_p into (2.5), we get the quadratic

$$aI_h^2 + bI_h + c = 0,$$

where

$$a = -\beta_{hh}N_{p0}(\mu_h+\Gamma_h)\frac{\gamma+\mu_h}{\gamma} < 0$$

$$b = \frac{N_{p0}(\gamma+\mu_h)(\mu_h+\Gamma_h)\Lambda_h}{\gamma}\left(\frac{\Lambda_h}{\mu_h} - \frac{N_{h0}}{R_3}\right) - \frac{\beta_{ph}I_{p2}^*N_{h0}(\mu_h+\Gamma_h)}{\gamma}$$

$$c = \Lambda_h\beta_{ph}I_{p2}^*N_{h0}$$

Solving the quadratic we get $I_h = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Since $a<0$, I_h will be non-negative if $-b\pm\sqrt{b^2-4ac} \le 0$.

When $-b+\sqrt{b^2-4ac} \le 0$

$\Rightarrow$ $\sqrt{b^2-4ac} \le b \Rightarrow b \ge 0$

$\Rightarrow$ $b^2 - 4ac \le b^2$

$\Rightarrow$ $-4ac \le 0$ which is not the case since $ac<0$.

$-b-\sqrt{b^2-4ac} \le 0$ when ever $b \ge 0$

When $b<0$, we have

$b^2 < b^2 - 4ac$

$\Rightarrow$ $0 \le -4\ ac$ which is true.

Therefore, $I_{h2}^* = \frac{-b-\sqrt{b^2-4ac}}{2a}$, , and $S_{h2}^* = \frac{N_{h0}N_{p0}(\mu_h+\gamma)(\mu_h+\Gamma)I_{h2}^*}{\gamma(N_{p0}\beta_{hh}I_{h2}^*+\beta_{ph}N_{h0}I_{p2}^*)}$. Hence we have and endemic equilibrium given by

$$EE_2 = \left(\frac{\Lambda_b}{\mu_b}, 0, \frac{N_{p0}}{R_2}, I^*_{p2}, S^*_{h2}, \frac{(\mu_h+\Gamma_h)I^*_{h2}}{\gamma}, I^*_{h2}, 0\right).$$

When $S_b = \frac{N_{b0}}{R_1}$, from (2.1), we get $I_b = I^*_{b3} = \frac{\Lambda_b}{\mu_b} - \frac{N_{b0}}{R_1} > 0$. From equation (2.8), we get $B = B^*_3 = \frac{\varepsilon_b I^*_{b3}}{\delta}$. Substituting B into (2.4), we get $S_p = \frac{N_{p0}(\mu_p+\Gamma_p)(\kappa+B^*_3)I_p}{\beta_{pp}(\kappa+B^*_3)I_p+\tau_p N_{p0}B^*_3}$. Substituting B and S_p into (2.3), we get $aI_p^2 + bI_p + c$, where

$a_1 = -2\beta_{pp}(\mu_p+\Gamma_p)(\kappa+B^*_3) < 0$, $b_1 = \Lambda_p\beta_{pp}(\kappa+B^*_3) - 2\tau_p B^*_3 N_{p0}(\mu_p+\Gamma_p)$, $c_1 = \Lambda_p N_{p0}\tau_p B^*_3 > 0$.

Since $a_1 < 0, c_1 > 1$ and we are not sure about the sign of b_1, the same analysis as done above can be used to conclude that $I^*_{p3} = \frac{-b_1-\sqrt{b_1^2-4a_1c_1}}{2a_1}$, with a, b and c taken as a_1, b_1 and c_1 respectively. Substituting I_p, B and E_h into (2.6), we get

$$S_h = \frac{N_{p0}Nz_{h0}(\gamma+\mu_h)(\mu_h+\Gamma_h)(\kappa+B^*_3)I_h}{\gamma[\beta_{hh}N_{p0}(\kappa+B^*_3)I_h+\tau_h B^*_3 N_{h0}N_{p0}+\beta_{ph}I^*_{p3}N_{h0}(\kappa+B^*_3)]}$$

Substituting B, S_h and I_p into (2.5), we get $a_2I_h^2 + b_2I_h + c_2 = 0$, where

$$a_2 = -\frac{\beta_{hh}(\gamma+\mu_h)(\mu_h+\Gamma_h)N_{p0}(\kappa+B^*_3)}{\gamma} < 0$$

$$b_2 = \Lambda_h\beta_{hh}N_{p0}(\kappa+B^*_3) - \frac{(\gamma+\mu_h)(\mu_h+\Gamma_h)(\kappa+B^*_3)N_{h0}N_{p0}}{\gamma}\left(\frac{\tau_h B^*_3}{\kappa+B^*_3} + \frac{\beta_{ph}I^*_{p3}}{N_{p0}+\mu_h}\right)$$

$$c_2 = \Lambda_h\tau_h B^*_3 N_{p0}N_{h0} + \beta_{hh}I^*_{p3}(\kappa+B^*_3)N_{h0} > 0$$

In the same way, here we have $a_2 < 0, c_2 > 0$ and we do not know the sign of b_2, thus we can make the same conclusion that $I^*_{h3} = \frac{-b_2-\sqrt{b_2^2-4a_2c_2}}{2a_2}$. Thus

$$EE_3 = \left(\frac{N_{b0}}{R_1}, I^*_{b3}, S^*_{p3}, I^*_{p3}, S^*_{h3}, E^*_{h3}, I^*_{h3}, \frac{\varepsilon_b I^*_{b3}}{\delta}\right).$$

Where $S^*_{p3} = \frac{N_{p0}(\mu_p+\Gamma_p)(\kappa+B^*_3)I^*_{p3}}{\beta_{pp}(\kappa+B^*_3)I^*_{p3}+\tau_p N_{p0}B^*_3}$,

$$S^*_{h3} = \frac{N_{p0}N_{h0}(\gamma+\mu_h)(\mu_h+\Gamma_h)(\kappa+B^*_3)I^*_{h3}}{\gamma[\beta_{hh}N_{p0}(\kappa+B^*_3)I^*_{h3}+\tau_h B^*_3 N_{h0}N_{p0}+\beta_{ph}I^*_{p3}N_{h0}(\kappa+B^*_3)]}, \quad E^*_{h3} = \frac{(\mu_h+\Gamma_h)I^*_{h3}}{\gamma}$$

3.2.1 Local stability of endemic equilibria

Lemma 3.4 (Routh Hurwitz criterion). [11] *The requirement for the polynomial* $s^3 + as^2 + bs + c = 0$ *to have negative real roots or complex roots with negative real parts is that*

(a) $a > 0$

(b) $ab - c > 0$, and

(c) $c > 0$

Theorem 3.5: *The endemic* EE_1, *where the infection is present in human population only, is locally stable when* $R_1 < 1, R_2 < 1$ *and* $R_3 > 1$ *and is unstable when* $R_1 > 1$ *or* $R_2 > 1$

Proof: From the characteristic equation $|J_{EE1} - eI_8| = 0$, where J_{EE1} is the Jacobian matrix evaluated at EE_1, we get

$$(-\mu_b - e)(\beta_{bb} - \mu_b - e)(-\mu_p - e)(-\delta - e)(\beta_{pp} - (\mu_p + \Gamma_p) - e) = 0 \tag{3.23}$$

or

$$e^3 + a_1 e^2 + b_1 e + c_1 = 0, \tag{3.24}$$

where

$$a_1 = \frac{\beta_{hh} I^*_{h1}}{N_{h0}} + 3\mu_h + \gamma + \Gamma_h, \qquad b_1 = (\Gamma_h + \gamma + 2\mu_h)\left(\frac{\beta_{hh} I^*_{h1}}{N_{h0}} + \mu_h\right), \qquad c_1 = \frac{\gamma \beta^2_{hh} I^*_{h1}}{R_3 N_{h0}}$$

From (3.23), all the other eigenvalues are negative. For factors $(\beta_{bb} - \mu_b - e)$ and $(\beta_{pp} - (\mu_p + \Gamma_p) - e)$, the eigenvalues are negative provided that $R_1 < 1$ and $R_2 < 1$, respectively.

We note from (3.24) that $a_1 > 0$ and $c_1 > 0$. We recall that $R_3 = \frac{\gamma \beta_{hh}}{(\gamma + \mu_h)(\mu_h + \Gamma_h)}$. It can be shown that

$$\begin{aligned} a_1 b_1 - c_1 &= \left(\frac{\beta_{hh} I^*_{h1}}{N_{h0}} + 3\mu_h + \gamma + \Gamma_h\right)(\Gamma_h + \gamma + 2\mu_h)\left(\frac{\beta_{hh} I^*_{h1}}{N_{h0}} + \mu_h\right) - \frac{\gamma \beta^2_{hh} I^*_{h1}}{R_3 N_{h0}} \\ &> \frac{\beta_{hh} I^*_{h1}}{N_{h0}}(\mu_h + \gamma)(\Gamma_h + \mu_h) - \frac{\beta_{hh} I^*_{h1}(\gamma + \mu_h)(\Gamma + \mu_h)}{N_{h0}} = 0 \end{aligned}$$

Therefore using lemma 3.4, and the result that the disease-free EE_0 is globally stable when $R_3 < 1$, and unstable when $R_3 > 1$, we conclude that EE_1 is locally stable when $R_1 < 1$, $R_2 < 1$ and $R_3 > 1$.

Theorem 3.6: *The endemic equilibrium* EE_2, *where the disease is present in pig and human population only, is locally stable when* $R_2 > 1$ *and* $R_1 < 1$, *and is unstable when* $R_1 > 1$.

Proof: From the characteristic equation $|J_{EE2} - eI_8| = 0$, using cofactor expansion, we get

$$(-\mu_b - e)(\beta_{bb} - \mu_b - e)(-\delta - e) = 0 \tag{3.25}$$

or

$$e^3 + a_2 e^2 + b_2 e + c_2 = 0, \tag{3.26}$$

where

$$a_2 = \frac{\beta_{hh} I^*_{h2}}{N_{h0}} + \frac{\beta_{ph} I^*_{p2}}{N_{p0}} + 3\mu_h + \gamma + \Gamma_h$$

$$b_2 = \left(\frac{\beta_{hh} I^*_{h2}}{N_{h0}} + \frac{\beta_{ph} I^*_{p2}}{N_{p0}} + \mu_h\right)(\gamma + 2\mu_h + \Gamma_h) + (\gamma + \mu_h)(\mu_h + \Gamma_h) - \frac{\gamma\beta_{hh} S^*_{h2}}{N_{h0}}$$

$$c_2 = \left(\frac{\beta_{hh} I^*_{h2}}{N_{h0}} + \frac{\beta_{ph} I^*_{p2}}{N_{p0}} + \mu_h\right)(\gamma + \mu_h)(\mu_h + \Gamma_h) - \frac{\gamma\mu_h\beta_{hh} S^*_{h2}}{N_{h0}},$$

or

$$e^2 + \left(\frac{\beta_{pp} I^*_{p2}}{N_{p0}} + \mu_p\right)e + \frac{\beta_{bb} I^*_{p2}}{N_{p0} R_2} = 0 \tag{3.27}$$

We recall that

$$\begin{aligned} S^*_{h2} &= \frac{N_{h0} N_{p0} (\gamma + \mu_h)(\mu_h + \Gamma_h) I^*_{h2}}{\gamma(N_{p0}\beta_{hh} I^*_{h2} + \beta_{ph} N_{h0} I^*_{p2})} \\ &< \frac{N_{h0} N_{p0} (\gamma + \mu_h)(\mu_h + \Gamma_h) I^*_{h2}}{\gamma N_{p0}\beta_{hh} I^*_{h2}} \\ &= \frac{N_{h0}(\gamma + \mu_h)(\mu_h + \Gamma_h)}{\gamma\beta_{hh}} \end{aligned} \tag{3.28}$$

We note from (3.26) that $a_2 > 0$ and that

$$\begin{aligned} c_2 &= \left(\frac{\beta_{hh} I^*_{h2}}{N_{h0}} + \frac{\beta_{ph} I^*_{p2}}{N_{p0}} + \mu_h\right)(\gamma + \mu_h)(\mu_h + \Gamma_h) - \frac{\gamma\mu_h\beta_{hh} S^*_{h2}}{N_{h0}} \\ &> \left(\frac{\beta_{hh} I^*_{h2}}{N_{h0}} + \frac{\beta_{ph} I^*_{p2}}{N_{p0}} + \mu_h\right)(\gamma + \mu_h)(\mu_h + \Gamma_h) - \frac{\gamma\mu_h\beta_{hh} N_{h0}(\gamma + \mu_h)(\mu_h + \Gamma_h)}{\gamma\beta_{hh} N_{h0}} \text{ by} \end{aligned} \tag{3.28}$$

$$= \left(\frac{\beta_{hh}I_{h2}^*}{N_{h0}} + \frac{\beta_{ph}I_{p2}^*}{N_{p0}} + \mu_h\right)(\gamma+\mu_h)(\mu_h+\Gamma_h) - \mu_h(\gamma+\mu_h)(\mu_h+\Gamma_h)$$
$$> \mu_h(\gamma+\mu_h)(\mu_h+\Gamma_h) - \mu_h(\gamma+\mu_h)(\mu_h+\Gamma_h) = 0.$$

Also

$$a_2b_2 - c_2 = (\mu_h+\Gamma_h)\left[\left(\frac{\beta_{hh}I_{h2}^*}{N_{h0}} + \frac{\beta_{ph}I_{p2}^*}{N_{p0}} + \mu_h\right)(\gamma+2\mu_h+\Gamma_h) + (\gamma+\mu_h)(\mu_h+\Gamma_h) - \right.$$
$$\left.\frac{\gamma\beta_{hh}S_{h2}^*}{N_{h0}}\right] - \left[\left(\frac{\beta_{hh}I_{h2}^*}{N_{h0}} + \frac{\beta_{ph}I_{p2}^*}{N_{p0}} + \mu_h\right)(\gamma+\mu_h)(\mu_h+\Gamma_h) - \frac{\gamma\mu_h\beta_{hh}S_{h2}^*}{N_{h0}}\right]$$
$$> (\mu_h+\Gamma_h)\left[\left(\frac{\beta_{hh}I_{h2}^*}{N_{h0}} + \frac{\beta_{ph}I_{p2}^*}{N_{p0}} + \mu_h\right)(\gamma+2\mu_h+\Gamma_h)\right] - \tag{3.28}$$
$$\left[\left(\frac{\beta_{hh}I_{h2}^*}{N_{h0}} + \frac{\beta_{ph}I_{p2}^*}{N_{p0}} + \mu_h\right)(\gamma+\mu_h)(\mu_h+\Gamma_h) - \frac{\gamma\mu_h\beta_{hh}S_{h2}^*}{N_{h0}}\right] \quad \text{using}$$
$$> \frac{\gamma\mu_h\beta_{hh}S_{h2}^*}{N_{h0}} > 0$$

Thus by lemma 3.4, all eigenvalues obtained from (3.26) are either real and negative, or complex with negative real parts.

From (3.27), using Descartes' rule of signs, we do not have positive real roots and we have at most two negative real roots. In the case where we have no real roots, the real parts of both complex roots are both equal to $-\frac{1}{2}\left(\frac{\beta_{pp}I_{p2}^*}{N_{p0}} + \mu_p\right)$, hence they are negative.

Therefore, when $R_2 > 1$ and $R_1 < 1$, EE_2 is locally stable.

Theorem 3.7: *The endemic equilibrium EE_3, where the disease is present in all the three populations, is locally stable when $R_1 > 1$ and is unstable when $R_1 < 1$.*

Proof: From the characteristic equation $|J_{EE3} - eI_8| = 0$, using cofactor expansion, we get

$$e^2 + \left(\mu_b + \frac{\beta_{bb}I_{b3}^*}{N_{b0}}\right)e + \mu_b^2 = 0 \tag{3.29}$$

or

$$e^2 + \left(\frac{\beta_{pp}I_{p3}^*}{N_{p0}} + \frac{\tau_pB_3^*}{\kappa+B_3^*} + \mu_p - \frac{\beta_{pp}S_{p3}^*}{N_{p0}} + d_p\right)e + d_p\left(\frac{\beta_{pp}I_{p3}^*}{N_{p0}} + \frac{\tau_pB_3^*}{\kappa+B_3^*} + \mu_p\right) - \mu_p\frac{\beta_{pp}S_{p3}^*}{N_{p0}} = 0 \tag{3.30}$$

or

$$e^3 + a_3e^2 + b_3e + c_3 = 0, \tag{3.31}$$

where

$$a_3 = \frac{\beta_{hh} I_{h3}^*}{N_{h0}} + \frac{\tau_h B_3^*}{\kappa + B_3^*} + \frac{\beta_{ph} I_{p3}^*}{N_{p0}} + 3\mu_h + \gamma + \Gamma_h$$

$$b_3 = \left(\frac{\beta_{hh} I_{h3}^*}{N_{h0}} + \frac{\tau_h B_3^*}{\kappa + B_3^*} + \frac{\beta_{ph} I_{p3}^*}{N_{p0}} + \mu_h \right)(\gamma + 2\mu_h + \Gamma_h) + (\mu_h + \Gamma_h)(\gamma + \mu_h) - \gamma \frac{\beta_{hh} S_{h3}^*}{N_{h0}}$$

$$c_3 = \left(\frac{\beta_{hh} I_{h3}^*}{N_{h0}} + \frac{\tau_h B_3^*}{\kappa + B_3^*} + \frac{\beta_{ph} I_{p3}^*}{N_{p0}} + \mu_h \right)(\mu_h + \Gamma_h)(\gamma + \mu_h) - \gamma \mu_h \frac{\beta_{hh} S_{h3}^*}{N_{h0}}$$

For (3.29), making similar analysis as we did for (3.27), we can deduce that all roots have negative real parts. Similarly, for (3.30), as soon as we have shown that the coefficient of e and the constant term are positive, the same analysis and conclusion can be made as we did for (3.27). We proceed as follows to show that the mentioned coefficients in (3.00) are indeed positive. We recall that

$$\begin{aligned} S_{p3}^* &= \frac{d_p N_{p0} (\kappa + B_3^*) I_{p3}^*}{\beta_{pp} (\kappa + B_3^*) I_{p3}^* + \tau_p N_{p0} B_3^*} \\ &< \frac{d_p N_{p0} (\kappa + B_3^*) I_{p3}^*}{\beta_{pp} (\kappa + B_3^*) I_{p3}^*} \\ &= \frac{d_p N_{p0}}{\beta_{pp}} \end{aligned} \tag{3.23}$$

Looking at the coefficient of e in (3.30)

$$\left(\frac{\beta_{pp} I_{p3}^*}{N_{p0}} + \frac{\tau_p B_3^*}{\kappa + B_3^*} + \mu_p - \frac{\beta_{pp} S_{p3}^*}{N_{p0}} + d_p \right) > \left(\frac{\beta_{pp} I_{p3}^*}{N_{p0}} + \frac{\tau_p B_3^*}{\kappa + B_3^*} + \mu_p - d_p + d_p \right) \quad \text{using}$$

$$= \frac{\beta_{pp} I_{p3}^*}{N_{p0}} + \frac{\tau_p B_3^*}{\kappa + B_3^*} + \mu_p > 0$$

For the constant coefficient in (3.30)

$$\begin{aligned} d_p \left(\frac{\beta_{pp} I_{p3}^*}{N_{p0}} + \frac{\tau_p B_3^*}{\kappa + B_3^*} + \mu_p \right) - \mu_p \frac{\beta_{pp} S_{p3}^*}{N_{p0}} &> d_p \left(\frac{\beta_{pp} I_{p3}^*}{N_{p0}} + \frac{\tau_p B_3^*}{\kappa + B_3^*} + \mu_p \right) - \mu_p d_p \quad \text{using} \\ &> d_p \mu_p - \mu_p d_p = 0 \end{aligned} \tag{3.32}$$

For equation (3.31), $a_3 > 0$. We recall that

$$\begin{aligned} S_{h3}^* &= \frac{(\gamma + \mu_h)(\mu_h + \Gamma_h) N_{h0} N_{p0} (\kappa + B_3^*) I_{h3}^*}{\gamma [\beta_{hh} N_{p0} (\kappa + B_3^*) I_{h3}^* + \tau_h B_3^* N_{h0} N_{p0} + \beta_{ph} I_{p3}^* N_{h0} (\kappa + B_3^*)]} \\ &< \frac{(\gamma + \mu_h)(\mu_h + \Gamma_h) N_{h0} N_{p0} (\kappa + B_3^*) I_{h3}^*}{\gamma \beta_{hh} N_{p0} (\kappa + B_3^*) I_{h3}^*} \end{aligned} \tag{3.33}$$

$$= \frac{(\gamma+\mu_h)(\mu_h+\Gamma_h)N_{h0}}{\gamma\beta_{hh}}$$

$$c_3 = \left(\frac{\beta_{hh}I^*_{h3}}{N_{h0}} + \frac{\tau_h B^*_3}{\kappa+B^*_3} + \frac{\beta_{ph}I^*_{p3}}{N_{p0}} + \mu_h\right)(\mu_h+\Gamma_h)(\gamma+\mu_h) - \gamma\mu_h\frac{\beta_{hh}S^*_{h3}}{N_{h0}}$$

$$> \left(\frac{\beta_{hh}I^*_{h3}}{N_{h0}} + \frac{\tau_h B^*_3}{\kappa+B^*_3} + \frac{\beta_{ph}I^*_{p3}}{N_{p0}} + \mu_h\right)(\mu_h+\Gamma_h)(\gamma+\mu_h) - \mu_h(\gamma+\mu_h)(\mu_h+\Gamma_h) \text{ using}$$

$$> \mu_h(\mu_h+\Gamma_h)(\gamma+\mu_h) - \mu_h(\gamma+\mu_h)(\mu_h+\Gamma_h) = 0 \qquad (3.33)$$

$$a_3b_3 - C_3 > (\gamma+\mu_h)\left[\left(\frac{\beta_{hh}I^*_{h3}}{N_{h0}} + \frac{\tau_h B^*_3}{\kappa+B^*_3} + \frac{\beta_{ph}I^*_{p3}}{N_{p0}} + \mu_h\right)(\gamma+2\mu_h+\Gamma_h) + (\mu_h+\Gamma_h)(\gamma+\mu_h) - \gamma\frac{\beta_{hh}S^*_{h3}}{N_{h0}}\right]$$

$$-\left(\left(\frac{\beta_{hh}I^*_{h3}}{N_{h0}} + \frac{\tau_h B^*_3}{\kappa+B^*_3} + \frac{\beta_{ph}I^*_{p3}}{N_{p0}} + \mu_h\right)(\mu_h+\Gamma_h)(\gamma+\mu_h) - \gamma\mu_h\frac{\beta_{hh}S^*_{h3}}{N_{h0}}\right)$$

$$> (\gamma+\mu_h)\left[\left(\frac{\beta_{hh}I^*_{h3}}{N_{h0}} + \frac{\tau_h B^*_3}{\kappa+B^*_3} + \frac{\beta_{ph}I^*_{p3}}{N_{p0}} + \mu_h\right)(\gamma+2\mu_h+\Gamma_h)\right] \text{ using}$$

$$-\left(\left(\frac{\beta_{hh}I^*_{h3}}{N_{h0}} + \frac{\tau_h B^*_3}{\kappa+B^*_3} + \frac{\beta_{ph}I^*_{p3}}{N_{p0}} + \mu_h\right)(\mu_h+\Gamma_h)(\gamma+\mu_h) - \gamma\mu_h\frac{\beta_{hh}S^*_{h3}}{N_{h0}}\right)$$

$$> \gamma\mu_h\frac{\beta_{hh}S^*_{h3}}{N_{h0}} = 0 \qquad (3.33)$$

Using Lemma 3.4, the roots of (3.31) also have negative real parts. Therefore when $R_1 > 1$, EE_3 is locally stable.

3.2.2 Global stability of endemic equilibria

Theorem 3.8: *The endemic equilibrium EE_1 is globally stable when $R_3 > 1$, $R_2 < 1$ and $R_1 < 1$.*

Proof: Equation (2.2) can be rearranged as follows

$$\frac{1}{\mu_b}\frac{I'_b}{I_b} = \frac{\beta_{bb}S_b}{\mu_b N_b} - 1$$

$$\Rightarrow \quad \frac{1}{\mu_b}\frac{I'_b}{I_b} = \frac{R_1 S_b}{N_b} - 1$$

$$\Rightarrow \quad \frac{1}{\mu_b}\frac{d}{dt}\ln(I_b) = \frac{R_1 S_b}{N_b} - 1$$

$$\Rightarrow \quad \frac{1}{\mu_b}\frac{d}{dt}\ln(I_b) < R_1 - 1 \qquad \text{since} \qquad \frac{S_b}{N_b} < 1$$

$$\Rightarrow \quad \frac{1}{\mu_b}\int_0^t \frac{d}{dt}\ln(I_b)dt < \int_0^t (R_1 - 1)dt$$

$$\Rightarrow \quad \frac{1}{\mu_b}\ln\left(\frac{I_b(t)}{I_b(0)}\right) < (R_1 - 1)t$$

$$\Rightarrow \quad \left(\frac{I_b(t)}{I_b(0)}\right)^{\frac{1}{\mu_b}} < e^{(R_1-1)t}$$

$$\Rightarrow \quad I_b(t) < I_b(0)e^{\mu_b(R_1-1)t}$$

$$\Rightarrow \quad \lim_{t\to\infty} I_b(t) < I_b(0)\lim_{t\to\infty} e^{\mu_b(R_1-1)t} \to 0 \qquad \text{when} \qquad R_1 < 1$$

Since I_b is bounded below by zero, we can conclude that the hyperplane $I_b = 0$ attracts all solutions of our model when $R_1 < 1$.

On the hyperplane $I_b = 0$, equation (2.8) becomes

$$B' = -\delta B$$

$$\frac{B'}{B} = -\delta$$

$$\int_0^t \frac{d}{dt}\ln(B)dt = \int_0^t -\delta dt$$

$$\ln\left(\frac{B(t)}{B(0)}\right) = -\delta t$$

$$\lim_{t\to\infty} B(t) = B(0)\lim_{t\to\infty} e^{-\delta t} = 0$$

Hence on the subspace defined by $I_b = 0$, the hyperplane $B = 0$ attracts all solutions of the model.

We have established that $I_b = 0$ and $B = 0$ are attractors when $R_1 < 1$. In this space, equation (2.4) becomes

$$I_p' = \frac{\beta_{pp}S_pI_p}{N_p} - d_pI_p$$

$$\Rightarrow \quad \beta_{pp}\frac{1}{d_p}\frac{I_p'}{I_p} = \frac{R_2S_p}{N_p} - 1$$

$$\Rightarrow \quad \frac{1}{d_p}\frac{d}{dt}\ln(I_p) = \frac{R_2S_p}{N_p} - 1$$

$$\Rightarrow \quad \frac{1}{d_p}\frac{d}{dt}\ln(I_p) < R_2 - 1 \qquad \text{since} \qquad \frac{S_p}{N_p} < 1$$

$$\Rightarrow \quad \frac{1}{d_p}\int_0^t \frac{d}{dt}\ln(I_p)dt < \int_0^t (R_2 - 1)dt$$

$$\Rightarrow \quad \frac{1}{d_p}\ln\left(\frac{I_p(t)}{I_p(0)}\right) < (R_2 - 1)t$$

$$\Rightarrow \quad \left(\frac{I_p(t)}{I_p(0)}\right)^{\frac{1}{d_p}} < e^{(R_2-1)t}$$

$$\Rightarrow \qquad I_p(t) < I_p(0)e^{d_p(R_2-1)t}$$

$$\Rightarrow \qquad \lim_{t\to\infty} I_p(t) < I_p(0)\lim_{t\to\infty} e^{d_p(R_2-1)t} \to 0 \qquad \text{when} \qquad R_2 < 1$$

Therefore when $R_1 < 1$ and $R_2 < 1$, the hyperplanes $I_b = 0$, $B = 0$ and $I_p = 0$ are attractors. In this subspace, our model reduces to

$$\begin{aligned} S_h' &= \Lambda_h - \beta_{hh}\frac{S_h I_h}{N_h} - \mu_h S_h \\ E_h' &= \beta_{hh}\frac{S_h I_h}{N_h} - (\gamma + \mu_h)E_h \\ I_h' &= \gamma E_h - d_h I_h \end{aligned} \tag{3.34}$$

For this model, we now use the Graph theoretic method described in [24] to prove the global stability of EE_1

Consider the functions

$$D_1 = S_h - S_h^* - S_h^* \ln\frac{S_h}{S_h^*}, \qquad D_2 = E_h - E_h^* - E_h^* \ln\frac{E_h}{E_h^*}, \qquad D_3 = I_h - I_h^* - I_h^* \ln\frac{I_h}{I_h^*}$$

Differentiating along the trajectories of model (3.34), we get

$$\begin{aligned} D_1' &= \left(\frac{S_h - S_h^*}{S_h}\right)S_h' \\ &= \left(\frac{S_h - S_h^*}{S_h}\right)\left(\beta_{hh}\frac{S_h^* I_h^*}{N_h} - \beta_{hh}\frac{S_h I_h}{N_h} + \mu_h S_h^* - \mu_h S_h\right) \\ &= -\mu_h\frac{(S_h - S_h^*)^2}{S_h} + \left(1 - \frac{S_h^*}{S_h}\right)\left(\beta_{hh}\frac{S_h^* I_h^*}{N_h} - \beta_{hh}\frac{S_h I_h}{N_h}\right) \\ &\le \beta_{hh}\frac{S_h^* I_h^*}{N_h}\left(\frac{I_h}{I_h^*} - \ln\frac{I_h}{I_h^*} - \frac{S_h I_h}{S_h^* I_h^*} + \ln\frac{S_h I_h}{S_h^* I_h^*}\right) = a_{13}G_{13}, \end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned} D_2' &= \left(\frac{E_h - E_h^*}{E_h}\right)E_h' \\ &\le \beta_{hh}\frac{S_h^* I_h^*}{N_h}\left(\frac{S_h I_h}{S_h^* I_h^*} - \ln\frac{S_h I_h}{S_h^* I_h^*} - \frac{E_h}{E_h^*} + \ln\frac{E_h}{E_h^*}\right) \\ &= a_{21}G_{21}, \end{aligned}$$

and

$$D_3' = \left(\frac{I_h - I_h^*}{I_h}\right)I_h'$$

$$\leq \gamma E_h^* \left(\frac{E_h}{E_h^*} - \ln \frac{E_h}{E_h^*} - \frac{I_h}{I_h^*} + \ln \frac{I_h}{I_h^*} \right)$$
$$= a_{32} G_{32},$$

where $a_{13} = a_{21} = \beta_{hh} \frac{S_h^* I_h^*}{N_h}$ and $a_{32} = \gamma E_h^*$.

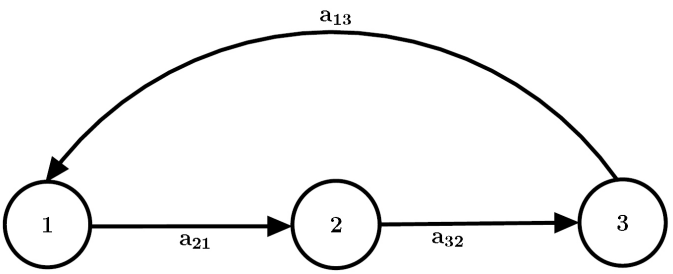

Figure 1

Digraph associated with model (3.34)

The associated weighted digraph has three vertices and one cycle (see Figure 1), along which $G_{13} + G_{21} + G_{32} = 0$. Therefore, by Theorem 3.5 of [24], there exist c_1, c_2 and c_3 such that the function $D = c_1 D1 + c_2 D_2 + c_3 D_3$ is a Lyapunov function for model 3.34 that can be used to prove global stability of EE_1 when $R_3 > 1, R_2 < 1$ and $R_1 < 1$.

Theorem 3.9: *The endemic equilibrium EE_2 is globally stable when $R_2 > 1$ and $R_1 < 1$*

Proof: When $R_1 < 1$, the hyperplanes $I_b = 0$ and $B = 0$ are still attractors and our initial model in this case reduces to .

$$\begin{aligned}
S_p' &= \Lambda_p - \beta_{pp} \frac{S_p I_p}{N_p} - \mu_p S_p \\
I_p' &= \beta_{pp} \frac{S_p I_p}{N_p} - d_p I_p \\
S_h' &= \Lambda_h - \beta_{hh} \frac{S_h I_h}{N_h} - \frac{\beta_{ph} I_p S_h}{N_p} - \mu_h S_h \\
E_h' &= \beta_{hh} \frac{S_h I_h}{N_h} + \frac{\beta_{ph} I_p S_h}{N_p} - (\gamma + \mu_h) E_h \\
I_h' &= \gamma E_h - d_h I_h
\end{aligned} \tag{3.35}$$

For this model, we again use the Graph theoretic method to prove global stability of EE_2 when $R_1 < 1$ and $R_2 > 1$. Consider functions $D_1 =$

$S_p - S_p^* - S_p^* \ln \frac{S_p}{S_p^*}$, $D_2 = I_p - I_p^* - I_p^* \ln \frac{I_p}{I_p^*}$, $D_3 = S_h - S_h^* - S_h^* \ln \frac{S_h}{S_h^*} + E_h - E_h^* - E_h^* \ln \frac{E_h}{E_h^*}$, and $D_4 = I_h - I_h^* - I_h^* \ln \frac{I_h}{I_h^*}$. Differentiating along the trajectories of model (3.35), it can be shown that we get

$$D_1' = \left(\frac{S_p - S_p^*}{S_p}\right) S_p' \le \beta_{pp} \frac{S_p^* I_p^*}{N_p} \left(\frac{I_p}{I_p^*} - \ln \frac{I_p}{I_p^*} - \frac{S_p I_p}{S_p^* I_p^*} + \ln \frac{S_p I_p}{S_p^* I_p^*}\right) = a_{12} G_{12},$$

$$D_2' = \left(\frac{I_p - I_p^*}{I_p}\right) I_p' \le \beta_{pp} \frac{S_p^* I_p^*}{N_p} \left(\frac{S_p I_p}{S_p^* I_p^*} - \ln \frac{S_p I_p}{S_p^* I_p^*} - \frac{I_p}{I_p^*} + \ln \frac{I_p}{I_p^*}\right) = a_{21} G_{21},$$

$$\begin{aligned} D_3' &= \left(\frac{S_h - S_h^*}{S_h}\right) S_h' + \left(\frac{E_h - E_h^*}{E_h}\right) E_h' \\ &\le \beta_{hh} \frac{S_h^* I_h^*}{N_h} \left(\frac{I_h}{I_h^*} - \ln \frac{I_h}{I_h^*} - \frac{E_h}{E_h^*} + \ln \frac{E_h}{E_h^*}\right) + \frac{\beta_{ph} I_p^* S_h^*}{N_p} \left(\frac{I_p}{I_p^*} - \ln \frac{I_p}{I_p^*} - \frac{E_h}{E_h^*} + \ln \frac{E_h}{E_h^*}\right) \\ &= a_{34} G_{34} + a_{32} G_{32}, \end{aligned}$$

and

$$D_4' = \left(\frac{I_h - I_h^*}{I_h}\right) I_h \le \gamma E_h^* \left(\frac{E_h}{E_h^*} - \ln \frac{E_h}{E_h^*} - \frac{I_h}{I_h^*} + \ln \frac{I_h}{I_h^*}\right) = a_{43} G_{43},$$

where $a_{12} = a_{21} = \beta_{pp} \frac{S_p^* I_p^*}{N_p}$, $a_{34} = \beta_{hh} \frac{S_h^* I_h^*}{N_h}$, $a_{32} = \frac{\beta_{ph} I_p^* S_h^*}{N_p}$ and $a_{43} = \gamma E_h^*$

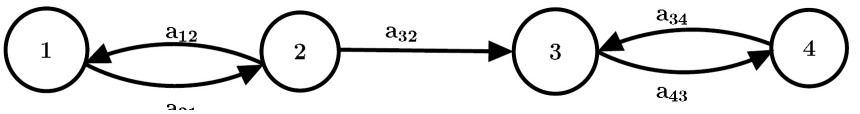

Figure 2

Digraph associated with model (3.35)

The associated digraph has four vertices and two cycles (See Figure 2), along which $G_{12} + G_{21} = 0$ and $G_{34} + G_{43} = 0$. Hence, by Theorem 3.5 of 24, there exist c_1, c_2, c_3 and c_4 such that the function $D = c_1 D1 + c_2 D_2 + c_3 D_3 + c_4 D_4$ is a Lyapunov function for model 3.35 that can be used to prove global stability of EE_2 when $R_3 > 1, R_2 > 1$ and $R_1 < 1$.

Theorem 3.10: *The endemic equilibrium* EE_3 *is globally stable when* $R_1 > 1, R_2 > 1$ *and* $R_3 > 1$.

Proof: Now considering our initial model, we study the global stability of EE_3 when $R_3 > 1$, $R_2 > 1$ and $R_1 > 1$.

Consider the following function

$$D_1 = S_b - S_b^* - S_b^* \ln \frac{S_b}{S_b^*}, \qquad D_2 = I_b - I_b^* - I_b^* \ln \frac{I_b}{I_b^*},$$

$$D_3 = S_p - S_p^* - S_p^* \ln \frac{S_p}{S_p^*} + I_p - I_p^* - I_p^* \ln \frac{I_p}{I_p^*} + \frac{\tau_p B^* S_p^*}{\varepsilon_b I_b^* (\kappa + B^*)} \left(B - B^* - B^* \ln \frac{B}{B^*} \right),$$

$$D_4 = S_h - S_h^* - S_h^* \ln \frac{S_h}{S_h^*} + E_h - E_h^* - E_h^* \ln \frac{E_h}{E_h^*} + \frac{\tau_h B^* S_h^*}{\varepsilon_b I_b^* (\kappa + B^*)} \left(B - B^* - B^* \ln \frac{B}{B^*} \right),$$

and

$$D_5 = I_h - I_h^* - I_h^* \ln \frac{I_h}{I_h^*}$$

Differentiating along the trajectories of our model, it can be shown that we get

$$D_1' = \left(\frac{S_b - S_b^*}{S_b} \right) S_b' \le \beta_{bb} \frac{S_b^* I_b^*}{N_b} \left(\frac{I_b}{I_b^*} - \ln \frac{I_b}{I_b^*} - \frac{S_b I_b}{S_b^* I_b^*} + \ln \frac{S_b I_b}{S_b^* I_b^*} \right) = a_{12} G_{12},$$

$$D_2' = \left(\frac{I_b - I_b^*}{I_b} \right) I_b' \le \beta_{bb} \frac{S_b^* I_b^*}{N_b} \left(\frac{S_b I_b}{S_b^* I_b^*} - \ln \frac{S_b I_b}{S_b^* I_b^*} - \frac{I_b}{I_b^*} + \ln \frac{I_b}{I_b^*} \right) = a_{21} G_{21}.$$

The quantity $\left(\frac{B(\kappa + B^*)}{B^*(\kappa + B)} - 1 \right) \left(1 - \frac{\kappa + B}{\kappa + B^*} \right)$ will be useful for our next calculations, we now show that this quantity is non-positive.

$$\begin{aligned} \left(\frac{B(\kappa + B^*)}{B^*(\kappa + B)} - 1 \right) \left(1 - \frac{\kappa + B}{\kappa + B^*} \right) &= - \left(\frac{B(\kappa + B^*)}{B^*(\kappa + B)} - 1 \right) \left(\frac{\kappa + B}{\kappa + B^*} - 1 \right) \\ &= - \frac{(B\kappa - B^*\kappa)(B - B^*)}{B^*(\kappa + B)(\kappa + B^*)} \\ &= - \frac{\kappa (B - B^*)^2}{B^*(\kappa + B)(\kappa + B^*)} \\ &\le 0. \end{aligned} \tag{3.36}$$

Differentiating the remaining functions along the trajectories of our model, we get

$$
\begin{aligned}
D_3' &= \left(\frac{S_p - S_p^*}{S_p}\right) S_p' + \left(\frac{I_p - I_p^*}{I_p}\right) I_p' + \frac{\tau_p B^* S_p^*}{\varepsilon_b I_b^* (\kappa + B^*)} \left(\frac{B - B^*}{B}\right) B' \\
&\leq \frac{\tau_p B^* S_p^*}{(\kappa + B^*)} \left(\left(\frac{B(\kappa + B^*)}{B^*(\kappa + B)} - 1\right)\left(1 - \frac{\kappa + B}{\kappa + B^*}\right) - \frac{I_p}{I_p^*} + \frac{I_b}{I_b^*} - \ln\frac{I_b}{I_b^*} + \ln\frac{I_p}{I_p^*}\right) \qquad (3.36) \\
&\leq \frac{\tau_p B^* S_p^*}{(\kappa + B^*)} \left(\frac{I_b}{I_b^*} - \ln\frac{I_b}{I_b^*} - \frac{I_p}{I_p^*} + \ln\frac{I_p}{I_p^*}\right) \qquad \text{using} \\
&= a_{32} G_{32},
\end{aligned}
$$

$$
\begin{aligned}
D_4' &= \left(\frac{S_h - S_h^*}{S_h}\right) S_h' + \left(\frac{E_h - E_h^*}{E_h}\right) E_h' + \frac{\tau_h B^* S_h^*}{\varepsilon_b I_b^* (\kappa + B^*)} \left(\frac{B - B^*}{B}\right) B' \\
&\leq \beta_{hh} \frac{S_h^* I_h^*}{N_h} \left(\frac{I_h}{I_h^*} - \ln\frac{I_h}{I_h^*} - \frac{E_h}{E_h^*} + \ln\frac{E_h}{E_h^*}\right) + \frac{\tau_h B^* S_h^*}{(\kappa + B^*)} \left(\frac{I_b}{I_b^*} - \ln\frac{I_b}{I_b^*} - \frac{E_h}{E_h^*} + \ln\frac{E_h}{E_h^*}\right) \\
&+ \beta_{ph} \frac{S_h^* I_p^*}{N_p} \left(\frac{I_p}{I_p^*} - \ln\frac{I_p}{I_p^*} - \frac{E_h}{E_h^*} + \ln\frac{E_h}{E_h^*}\right) = a_{45} G_{45} + a_{42} G_{42} + a_{43} G_{43},
\end{aligned}
$$

and

$$
D_5' = \left(\frac{I_h - I_h^*}{I_h}\right) I_h' \leq \gamma E_h^* \left(\frac{E_h}{E_h^*} - \ln\frac{E_h}{E_h^*} - \frac{I_h}{I_h^*} + \ln\frac{I_h}{I_h^*}\right) = a_{54} G_{54},
$$

where $a_{12} = a_{21} = \beta_{bb} \frac{S_b^* I_b^*}{N_b}$, $a_{32} = \frac{\tau_p B^* S_p^*}{\kappa + B^*}$, $a_{42} = \frac{\tau_h B^* S_h^*}{\kappa + B^*}$, $a_{43} = \beta_{ph} \frac{S_h^* I_p^*}{N_p}$, $a_{45} = \beta_{hh} \frac{S_h^* I_h^*}{N_h}$ and $a_{54} = \gamma E_h^*$

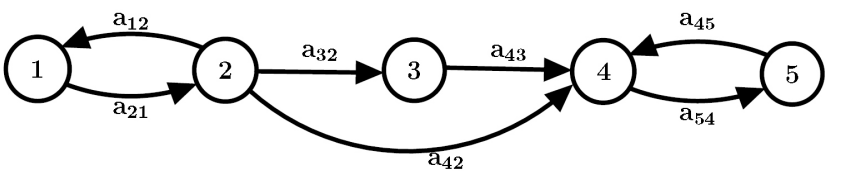

Figure 3

Digraph associated with model our initial model

The digraph associated with our initial model has five vertices and two cycles (see Figure 3), along which $G_{12} + G_{21} = 0$ and $G_{45} + G_{54} = 0$. Therefore, by Theorem 3.5 of [24], there exist c_1, c_2, c_3, c_4 and c_5 such that the function $D = c_1 D1 + c_2 D_2 + c_3 D_3 + c_4 D_4 + c_5 D_5$ is a Lyapunov function for our model that can be used to prove global stability of EE_3 when $R_3 > 1$, $R_2 > 1$ and $R_1 > 1$.

4. Numerical simulations

The initial population of study is assumed to be 1000 humans, 214 pigs and 1000 bats. The incidence rate in bats is not known, but based on observation, it is likely to be small [7], therefore it is assumed to be 0.0016. Prevalence rate is less than 5% [12]. The bat to bat transmission rate is calculated as $\beta_{bb} = \frac{\text{incidence}}{\text{prevalence}} = \frac{0.0016}{0.04} = 0.04$. In the review of medical and pathology records of 161 Livingstone's fruit bat Pteropus (LFBS) between 1992 and 2017, 88 deaths were recorded [20]. Based on this the natural bat death rate is estimated as $\mu_b = \frac{88}{161(2017-1992)} \approx 0.021$. Nipah virus can survive up to three days in some fruit juices, and at least seven days in date pas kept at $22^{\circ}c$ [2]. The rate at which virus loses virulence is therefore calculated as $\delta = \frac{1}{7} = 0.143$. Generally, fatality rate is low in pigs, except in piglets. The fatality rate can be high (40%) or low (<5%) [10]. We take disease induced death rate in pigs as $\Gamma_p = 0.4$. The case fatality in humans rages from 9% to 100% [1], thus $\Gamma_h = 0.95$. The incubation period ranges from 4 to 14 days, thus the rate at which exposed humans become infectious is estimated as $\gamma = \frac{1}{10} = 0.1$. In the review of 122 Nipah patients in Bangladesh, 7% of the patients to 62 other people (51%), who developed illness after 5 to 15 days of close contact [26]. The person to person transmission rate is estimated as $\beta_{hh} = \frac{0.51}{0.07} \approx 7.2$. According to Trading Economics, the death rate in South Asia was reported as 8.737% in 2021. The death rate of humans is estimated as $\mu_h = 08737$. The parameter value used for simulations are presented in Table 1 below.

Table 1

Parameter values

Parameter	Description	Parameter value
Λ_b	bat recruitment rate	21 [assumed]
μ_b	bat natural death rate	0.021 [20]
β_{bb}	bat contact rate	0.04 [7, 12]
δ	rate at which virus looses virulence	0.143 [2]
κ	Michaelis Menten constant	0.000542 [assumed]
ε_b	rate at which bats shed virus on the environment	0.005 [Assumed]
Γ_p	disease induced pig death rate	0.4 [10]

Contd...

μ_p	pig natural death rate	0.133 [assumed]
Λ_p	pig recruitment rate	60 [Assumed]
τ_p	the rate at which pigs ingest virus	0.001 [Assumed]
β_{pp}	pigs contact rate	1.8 [Assumed]
β_{ph}	pigs to human contact rate	0.001 [Assumed]
Γ_h	disease induced human death rate	0.95 [1]
μ_h	human natural death rate	0.08737 [calculated]
γ	rate at which exposed humans become infectious	0.1 [6, 9, 15, 23, 30]
Λ_h	the human recruitment rate	87 [Assumed]
β_{hh}	human to human contact rate	7.2 [26]
τ_h	rate at which humans ingest virus	0.00001 [Assumed]

With the above figures, the partial reproduction numbers are as follows:

$$R_1 \approx 1.9, \quad R_2 \approx 3.4, \quad R_3 \approx 3.7.$$

We have EE_3 below.

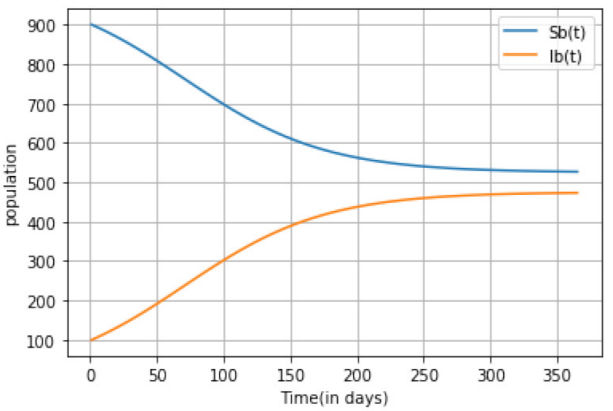

Figure 4

The approximated equilibrium values are $S_b^* = 540, I_b^* = 460$ **and** $\mathcal{R}_0 = 3.7 > 1$

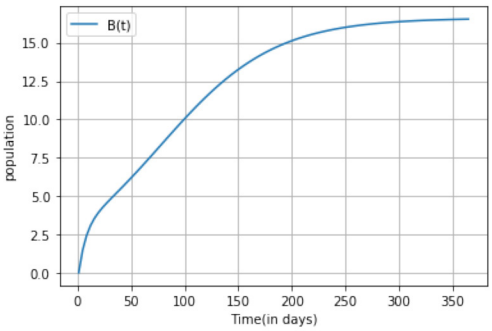

Figure 5

The approximated equilibrium value is $B = 16.5$ **and** $\mathcal{R}_0 = 3.7 > 1$

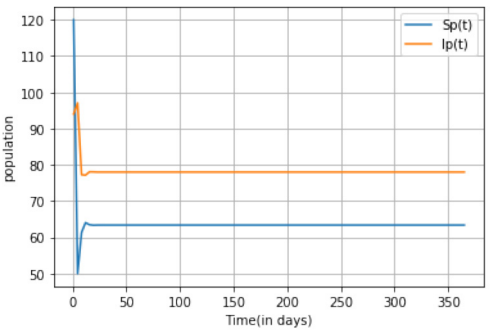

Figure 6

The approximated equilibrium values are $S_p^* = 65, I_p^* = 78$ **and** $\mathcal{R}_0 = 3.7 > 1$

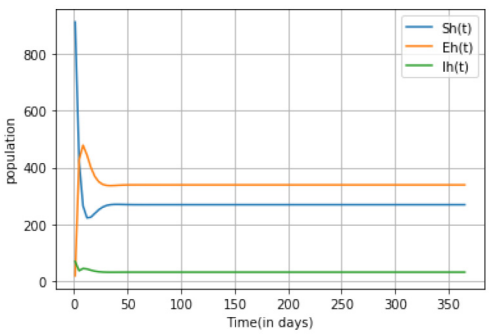

Figure 7

The approximated equilibrium values are $S_h^* = 270, E_h^* = 350, I_h^* = 50$ **and** $\mathcal{R}_0 = 3.7 > 1$

Setting $\beta_{bb} = 0.004$ gives $R_1 = 0.2 < 1$ We get EE_2

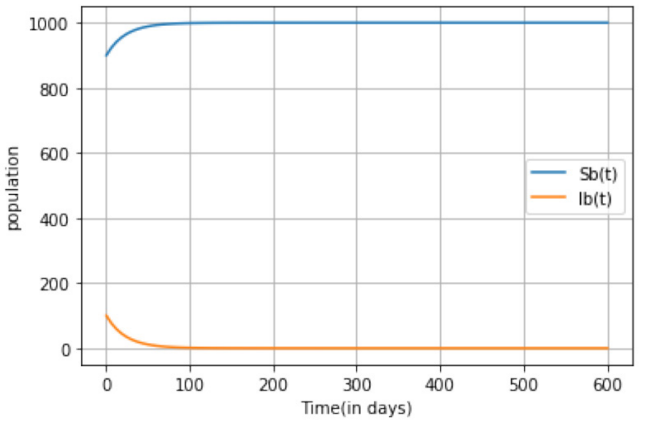

Figure 8

The approximated equilibrium values are $S_b^* = 1000, I_b^* = 0$ and $\mathcal{R}_0 = 3.7 > 1$

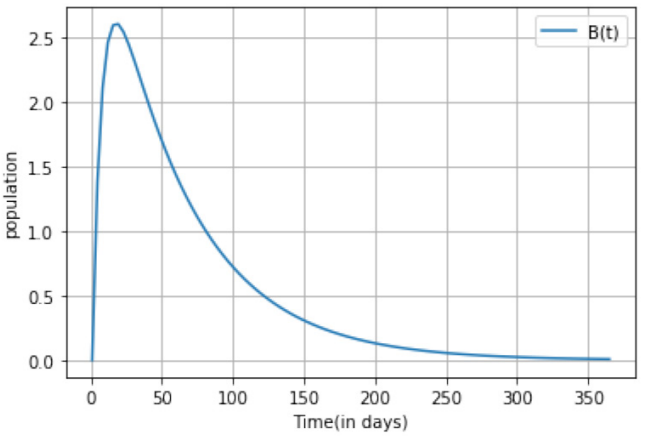

Figure 9

The approximated equilibrium value is $B = 0$ and $\mathcal{R}_0 = 3.7 > 1$

The equilibrium values of pig and human populations remain the same as in figures 6 and 7 respectively.

Keeping R_1 below one and setting $\beta_{pp} = 0.15$ gives $R_2 = 0.3 < 1$ and we get EE_1.

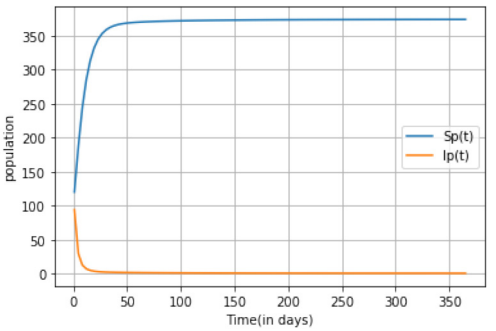

Figure 10

The approximated equilibrium values are $S_p^* = 375, I_p^* = 0$ **and** $\mathcal{R}_0 = 3.7 > 1$

The equilibrium values of compartment B, bat population and human population remain the same as in figures 8, 9 and 10 respectively.

Keeping R_1 and R_2 below one and setting $\beta_h h = 0.72$ gives $R_3 = 0.4 < 1$. We get the DFE E_0.

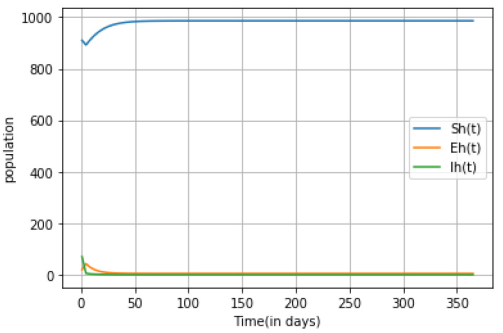

Figure 11

The approximated equilibrium values are $S_h^* = 999, E_h^* = 0, I_h^* = 0$ **and** $\mathcal{R}_0 = 0.4 < 1$

The equilibrium values of compartment B, bat population and pig population remain the same as in figures 8, 9 and 10 respectively.

5. Discussion and Conclusion

The model has four equilibrium points, the disease-free equilibrium demonstrated by figures 8, 9,10, 11 and three endemic equilibria. The first endemic equilibrium EE1 is depicted by figures 8, 9, 10, 7, EE2 is demonstrated by figures 8, 9,6, 7, while EE3 is represented by figures 4, 5,

6, 7. The partial basic reproduction numbers R_1, R_2 and R_3 estimate the number of secondary infections in bats, pigs and humans population, respectively, which happen when an infectious bat, pig or human is introduced into a susceptible population of the same specie. The basic reproduction number of the model is taken as the maximum of the three partial basic reproduction numbers. It is established that if $\mathcal{R}_0 = \max\{R_1, R_2, R_3\} = R_3$, the DFE is globally stable when $R_3 < 1$ and is unstable if $R_3 > 1$. It is demonstrated that if $R_1 < 1$, $R_2 < 1$ and $R_3 > 1$, the endemic equilibrium EE_1, where the infection is present only in the human population, is globally stable. If $R_1 < 1$, $R_2 > 1$ and $R_3 > 1$, the endemic equilibrium EE_2, where the infection is present in the human and pig populations, is globally stable. Lastly, when $R_1 > 1$, $R_2 < 1$ and $R_3 > 1$, the endemic equilibrium EE_3, where the infection in bat, pig and human population, is globally stable. Though the bats are natural hosts, we may have a case where the infection does not persist in bats, but persists in pigs or humans. For instance, it can happen that an infectious bat contaminates food consumed by pigs or humans. If the basic reproduction number in bats is less that 1, then the infection will not persist in bats. However, if the basic reproduction number in pigs is greater than one, the disease will persist in pigs and hence in humans. As per our model, since humans get infection from both bats and pigs, we cannot have a case where the disease persists in bats or pigs but not in humans, but the disease can persist in human population but die out in bats and pigs. Similarly, we cannot have a case where the disease persists in bats, but dies out in pigs or humans, but we can have a case where the disease dies out in bats but persists in pigs, hence in humans as well. To control Nipah virus outbreak, measures such as culling of bats or pigs and isolation of exposed humans for at least 14 days can be considered.

Appendix

A. *Appendix*

To calculate the basic reproduction number $\mathcal{R}_0$, we use the next generation matrix method. As described in [24], $\mathcal{R}_0 = \rho(FV^{-1}) = \max\{R_1, R_2, R_3\}$ where

$$F = \begin{bmatrix} \beta_{bb} & 0 & 0 & 0 & 0 \\ 0 & \beta_{pp} & 0 & 0 & \frac{\tau_p \Lambda_p}{\kappa \mu_p} \\ 0 & \frac{\beta_{pp} \Lambda_h \mu_p}{\mu_h \Lambda_p} & 0 & \beta_{hh} & \frac{\tau_h \Lambda_h}{\kappa \mu_h} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{A.1}$$

and

$$V = \begin{bmatrix} \mu_b & 0 & 0 & 0 & 0 \\ 0 & \Gamma_p + \mu_p & 0 & 0 & 0 \\ 0 & 0 & \gamma + \mu_h & 0 & 0 \\ 0 & 0 & -\gamma & \mu_h + \Gamma_h & 0 \\ -\varepsilon_b & 0 & 0 & 0 & \delta \end{bmatrix}. \tag{A.2}$$

The partial basic reproduction numbers R_1, R_2 and R_3 give the estimated number of secondary infections in each population specie, so

$$R_1 = \frac{\beta_{bb}}{\mu_b} \tag{A.3}$$

represents the basic reproduction number in bat population,

$$R_2 = \frac{\beta_{pp}}{\mu_p + \Gamma_p} \tag{A.4}$$

represents the basic reproduction number in pig population, and

$$R_3 = \frac{\gamma \beta_{hh}}{(\mu_h + \Gamma_h)(\gamma + \mu_h)} \tag{A.5}$$

represents the basic reproduction number in human population.

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