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M_n – Polynomials of some special for cog-graphs

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Abstract

The maximum distance between two subsets $\acute{S}$ and S of vertex set $V(G)$ of a connected graph G is maximum distance between any two vertices u and v such that u belong to $\acute{S}$ and v belong to S. In this paper, we take special case of maximum distance when $\acute{S}$ consist of one vertex and S consist of $(n-1)$ vertices, $n \geq 3$,. This distance is defined by :

$$d_{\max}(u,S) = \max\{d(u,v) : v \in S\}, |S| = n-1, 3 \leq n \leq p, u \in V(G),$$

Where p is the order of a graph G.

We founded M_n – polynomials, M_n – index, Hosoya polynomial and Wiener index for some special graphs such as : cog-complete, cog- star, cog-wheel, cog-path and cog-cycle.

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1. Introduction

The first to define the maximum distance between two subsets of vertex set are Dankelmann et al. in 1999 [1], as follows :

$d_{\max}(S', S) = \max\{d(u, v) : u \in S', v \in S\}$, where Ś and S are not necessarily distinct subsets of $V(G)$, and $d(u, v)$ is the number of edges in a shortest path between u and v. We note that $d_{\max}(S', S) = 0$ if $S' = S = \{w\}$; thus for $S' = S$ and $|S'|$ or $|S| \geq 2$, this implies that $d_{\max}(S', S) > 0$.

The ***special type of maximum distance*** between two subset of $V(G)$ such that the first subset Ś consists of one vertex and the second subset S consists of $(n-1)$ vertices, $n \geq 3,$. This distance is defined by :

$$d_{\max}(u, S) = \max\{d(u, v) : v \in S\}, |S| = n-1, 3 \leq n \leq p, u \in V(G),$$

Where p is the order of a graph G.

It is clear that for any vertex u and for any subset S of $V(G)$, we have : $d_{\max}(u, S) \geq 1$. If the vertex u dominates all vertices S, then $d_{\max}(u, S) = 1$.

When $n = 2$, the ***ordinary distance*** between two vertices of $V(G)$, can be obtained, [2,3]. Therefore, we let $n \geq 3$. To learn more about characteristics of this type of distance (see [4,5]).

The ***M_n - index of a graph G*** of order p, where, $3 \leq n \leq p$, is the sum of max – n– distances of all pairs (u, S) in G, i.e.

$$M_n(G) = \sum_{\substack{u \in V-S \\ S \subseteq V}} d_{\max}(u, S).$$

The ***M_n - polynomial of G*** of order p, is denoted by $\boldsymbol{M_n(G; x)}$, and defined by :

$M_n(G; x) = \sum_{k=m}^{\delta} C_n(G, k)x^k$, where δ is the diameter of G, where $m = \min\{d_{\max}(u, S), u \in V - S, S \subseteq V\}$ and $\boldsymbol{C_n(G, k)}$ is the number of pairs (u, S), $S \subseteq V(G)$, $|S| = n-1$, $3 \leq n \leq p$, such that $d_{\max}(u, S) = k$, for each $m \leq k \leq \delta$. It is clear that the $M_n -$ index of any graph G can be obtained from $M_n -$ polynomial as follows:

$$M_n(G) = \frac{d}{dx} M_n(G; x)|_{x=1} = \sum_{k=m}^{\delta} kC_n(G, k).$$

If $C_n(u, G, k)$ represents the number of pairs (u, S) such that $d_{\max}(u, S) = k$, then $C_n(G, k) = \sum_{u \in V(G)} C_n(u, G, k)$. From clearly that: $\sum_{k=1}^{\delta} C_n(G, k) = p\binom{p-1}{n-1}$ and $C_n(G, 1) = \sum_{\forall v \in V(G)} \binom{degv}{n-1}$, for all $3 \leq n \leq p$.

There are many recent studies on $D-$distance and detour number in graph theory (see [6, 7, 8]). For additional information about the applications for some types of these distances, (see[9,10]).

The **Hosoya polynomial** connected graph G is defined as follows [11,12]: $H(G;x)=\sum_{k=1}^{\delta} d(G,k)x^k$, where $\boldsymbol{d(G,\ k)}$ is the number of pairs unordered of distinct vertices that are at a distance k, and **Wiener index** of G is defined as:

$$W(G)=\frac{d}{dx}H(G;x)\big|_{x=1}=\sum_{k=1}^{\delta} kd\,(G,k).$$

Finally, there is great interest in many polynomials such as Schultz and Modified Schultz, (see [13]).

2. Cog-Complete Graph :

Definition : A cog-complete graph K_p^c is the graph constructed from a complete graph Kp, $p\geq 3$ of vertex set $V(K_p)=\{v_1,v_2,\ldots,v_p\}$ with p additional vertices $U=\{u_1,u_2,\ldots,u_p\}$, and $2p$ edges $\{u_iv_i,u_iv_{i+1}:i=1,2,\ldots,p\}$, $v_{p+1}\equiv v_1$.

Some Properties of A cog-complete Graph K_p^c :

1. **The order and size :** $p(K_p^c)=2p$ and $q(K_p^c)=\frac{p(p+3)}{2}$.
2. **The max -n- diameter :** $\delta(K_p^c)=3$, for all $p\geq 4$.
3. M_n **– polynomial of a vertex :** The vertices v_i, i = 1,2, …, p have the same the M_n- polynomial of a vertex in a graph K_p^c. Also, the vertices u_i, i = 1,2, …, p have the same the M_n- polynomial of a vertex in a graph K_p^c.
4. **The max – n – distance :** The vertices of the graph K_p^c have the max – n – distance between the vertex v and any subset S of vertices of $V(K_p^c)$, $|S|=n-1$, $3\leq n\leq 2p$, as following :

$$d_{\max}(v_i,S)\leq 2,\ \text{for all }\ i=1,2,\ldots,p,$$

$$d_{\max}(u_i,S)\leq 3,\ \text{for all }\ i=1,2,\ldots,p.$$

Theorem 2.1 : *For all* $p\geq 4$, *we have*

$$M_n(K_p^c;x)=p\left[\binom{p+1}{n-1}+\binom{2}{n-1}\right]x+p\left[\binom{2p-1}{n-1}+\binom{p+1}{n-2}-\binom{2}{n-1}\right]x^2$$

$$+p\left[\binom{2p-1}{n-1}-\binom{p+2}{n-1}\right]x^3.$$

Proof **:** Obvious. □

Corollary 2.2 : *For all* $p \geq 4$, *we have :*

$$M_n(K_p^c)=p\left[5\binom{2p-1}{n-1}-\binom{p+2}{n-1}-\binom{p+1}{n-1}-\binom{2}{n-1}\right].$$

□

3. Cog-Star Graph

Definition : A cog-star graph S_p^c is the graph constructed from a star graph S_p, $p \geq 4$, of vertex set $V(S_p)=\{v_1,v_2,\ldots,v_p\}$ with $(p-1)$ additional vertices $U=\{u_1,u_2,\ldots,u_{p-1}\}$, and $2(p-1)$ edges $\{u_iv_{i+1},u_iv_{i+2} : i=1,2,\ldots,p-1\}$, $v_{p+1}\equiv v_2$.

Some Properties of A cog-star Graph S_p^c **:**

1. **The order and size :** $p(S_p^c)=2p-1$ and $q(S_p^c)=3(p-1)$.
2. **The max** $-n-$ **diameter :** $\delta(S_p^c)=4$, for all $p\geq 5$.
3. M_n **– polynomial of a vertex :** The vertices v_i, i = 2,3, …, p have the same the M_n – polynomial of a vertex in a graph S_p^c. Also, the vertices u_i, $i=1,2,\ldots,p-1$ have the same the M_n – polynomial of a vertex in a graph S_p^c.
4. **The max – n – distance :** The vertices of the graph S_p^c have the max – n – distance between the vertex v and any subset S of vertices of $V(S_p^c)$, $|S| = n-1$, $3\leq n\leq 2p-1$, as following :

$$d_{\max}(v_1,S)\leq 2,$$

$$d_{\max}(v_i,S)\leq 3, \text{ for all } i=2,3,\ldots,p,$$

$$d_{\max}(u_i,S)\leq 4, \text{ for all } i=1,2,\ldots,p-1.$$

Theorem 3.1 : *For all* $p \geq 5,$ *we have*

$$M_n(S_p^c; x) = \sum\nolimits_{k=1}^{4} C_n(S_p^c, k)x^k, \text{ where}$$

$$C_n(S_p^c, 1) = \binom{p-1}{n-1} + (p-1)\binom{3}{n-1} + (p-1)\binom{2}{n-1},$$

$$C_n(S_p^c, 2) = (p-1)\left[\binom{p+1}{n-1} + \binom{5}{n-1} - \binom{3}{n-1} - \binom{2}{n-1}\right] + \binom{2p-2}{n-1} - \binom{p-1}{n-1}.$$

$$C_n(S_p^c, 3) = (p-1)\left[\binom{2p-2}{n-1} + \binom{p+1}{n-2} - \binom{5}{n-1}\right].$$

$$C_n(S_p^c, 4) = (p-1)\left[\binom{2p-2}{n-1} - \binom{p+2}{n-1}\right].$$

***Proof* :** Obvious □

Corollary 3.2 : *For all* $p \geq 5,$ *we have*

$$M_n(S_p^c) = (7p-5)\binom{2p-2}{n-1} - \binom{p-1}{n-1} - (p-1)\left[\binom{p+2}{n-1} + \binom{p+1}{n-1} + \binom{5}{n-1} + \binom{3}{n-1} + \binom{2}{n-1}\right].$$

□

4. Cog-Wheel Graph :

Definition : A cog-wheel graph W_p^c is the graph constructed from a wheel *Wp*, $p \geq 5,$ of vertex set $V(W_p) = \{v_1, v_2, \ldots, v_p\},$ and with $(p - 1)$ additional vertices $U = \{u_1, u_2, \ldots, u_{p-1}\},$ and $2(p - 1)$ edges $\{u_i v_{i+1}, u_i v_{i+2} : i = 1, 2, \ldots, p-1\},\ v_{p+1} \equiv v_2.$

Some Properties of A Cog-Wheel Graph W_p^c :

1. **The order and size :** $p(W_p^c) = 2p-1$ and $q(W_p^c) = 4(p-1).$
2. **The max -*n*- diameter :** $\delta(W_p^c) = 4,$ for all $p \geq 7.$

3. $\boldsymbol{M_n}$ **– polynomial of a vertex :** The vertices v_i, $i = 2,3, \ldots, p$ have the same the $M_n -$ polynomial of a vertex in a graph W_p^c. Also, the vertices u_i, $i = 1,2,\ldots,p-1$ have the same the $M_n -$ polynomial of a vertex in a graph W_p^c.
4. **The max – n – distance :** The vertices of the graph W_p^c have the max – n – distance between the vertex v and any subset S of vertices of $V(W_p^c)$, $|S| = n-1$, $3 \le n \le 2p-1$, as following :

$$d_{\max}(v_1, S) \le 2,$$

$$d_{\max}(v_i, S) \le 3, \text{ for all } i = 2, 3, \ldots, p,$$

$$d_{\max}(u_i, S) \le 4, \text{ for all } i = 1, 2, \ldots, p-1.$$

Theorem 4.1 : *For all* $p \ge 7$, *we have*

$$M_n(W_p^c; x) = \sum_{k=1}^{4} C_n(W_p^c, k)x^k, \text{ where}$$

$$C_n(W_p^c, 1) = \binom{p-1}{n-1} + (p-1)\binom{5}{n-1} + (p-1)\binom{2}{n-1},$$

$$C_n(W_p^c, 2) = (p-1)\left[\binom{p+3}{n-1} + \binom{7}{n-1} - \binom{5}{n-1} - \binom{2}{n-1}\right] + \binom{2p-2}{n-1} - \binom{p-1}{n-1},$$

$$C_n(W_p^c, 3) = (p-1)\left[\binom{2p-2}{n-1} + \binom{p+3}{n-2} - \binom{7}{n-1}\right],$$

$$C_n(W_p^c, 4) = (p-1)\left[\binom{2p-2}{n-1} - \binom{p+4}{n-1}\right].$$

Proof **:** Obvious. □

Corollary 4.2 : *For all* $p \ge 7$, *we have*

$$M_n(W_p^c) = (7p-5)\binom{2p-2}{n-1} - \binom{p-1}{n-1} - (p-1)\left[\binom{p+4}{n-1} + \binom{p+3}{n-1}\right.$$

$$+\binom{7}{n-1}+\binom{5}{n-1}+\binom{2}{n-1}\Bigg].$$

□

5. Cog - Path Graph

Definition : A cog - path graph P_p^c is the graph constructed from a path P_p, $p \geq 3$, of set vertex $V(P_p)=\{v_1, v_2, \ldots, v_p\}$, and with $(p-1)$ additional vertices $U=\{u_1, u_2, \ldots, u_{p-1}\}$, and $2(p-1)$ edges $\{u_i v_i, u_i v_{i+1} : i=1,2,\ldots,p-1\}$.

Some Properties of A Cog-Path Graph P_p^c :

1. **The order and the size :** $p(P_p^c)=2p-1$ and $q(P_p^c)=3(p-1)$.
2. **The max -n- diameter :** $\delta(P_p^c)=p-1$, for all $p \geq 3$.
3. M_n **– polynomial of a vertex :** For all $1 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor$, then,
    - $M_n(v_i, P_p^c; x)=M_n(v_{p-i+1}, P_p^c; x)$.
    - $M_n(u_i, P_p^c; x)=M_n(u_{p-i}, P_p^c; x)$.
4. **The max – n – distance :** The vertices of the graph P_p^c have the max – n – distance between the vertex v and any subset S of vertices of $V(P_p^c)$, $|S| = n-1$, $3 \leq n \leq 2p-1$, as following :

$$d_{\max}(u_i, S)=d_{\max}(v_i, S) \leq p-i, \text{ for all } 1 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor.$$

Theorem 5.1 : *Let* P_p^c *be a cog-path graph of order* $2p-1$, $p \geq 4$. *Then for all* $3 \leq n \leq 2p-1$, *we have*:

$$M_n(P_p^c; x)=\sum_{k=1}^{p-1} C_n(P_p^c, k)x^k \text{, where}$$

$$C_n(P_p^c, 1)=(p+1)\binom{2}{n-1}+(p-2)\binom{4}{n-1},$$

$$C_n(P_p^c, k)=\binom{p-2k}{1}\left\{\binom{4k}{n-1}-\binom{4k-4}{n-1}\right\}+\binom{p-2k+1}{1}\left\{\binom{4k-2}{n-1}-\binom{4k-6}{n-1}\right\}$$

$$+\begin{cases} 2\binom{4k-4}{n-1}+2\binom{4k-2}{n-1}-4\binom{2k-2}{n-1}; 2\le k\le \left\lfloor \frac{p}{2}\right\rfloor, \\ 4\binom{2p-2}{n-1}-4\binom{2k-2}{n-1}; \left\lfloor \frac{p}{2}\right\rfloor+1\le k\le p-1. \end{cases}$$

Proof **:** It is obvious that $C_n(P_p^c,1)=(p+1)\binom{2}{n-1}+(p-2)\binom{4}{n-1}$. To find $C_n(P_p^c,k)$ for all $2\le k\le p-1$, $p\ge 6$, let $U\cup V$ be the set of vertices of P_p^c, where $U=\{u_i:i=1,2,\dots,p-1\}$ and $V=\{v_i:i=1,2,\dots,p\}$ and let S be a subset of vertices of P_p^c has length $n-1, n\ge 3$. Then, there are three cases :

Case I : For all $2\le k\le \left\lfloor \frac{p}{2}\right\rfloor-1$, then there are two subcases:

a. There are two vertices v_{i+k}, u_{i+k-1} lying at a distance k from v_i, for all $1\le i\le k$ and there are $2k+2i-4$ vertices $S_1\cup\{v_{i+1},\dots,v_{i+k-1}\}\cup\{u_1,\dots,u_{i+k-2}\}$, where $S_1=\{v_1,\dots,v_{i-1}\}$, (if i = 1, then $S_1=\emptyset$) lying at a distance less than k to v_i, for all $1\le i\le k$, also, there are two vertices v_{i+k}, u_{i+k-1} lying at a distance k from u_i, for all $1\le i\le k-1$ and there are $2k+2i-4$ vertices $S_2\cup\{u_{i+1},\dots,u_{i+k-2}\}\cup\{v_1,\dots,v_{i+k-1}\}$, where $S_2=\{u_1,\dots,u_{i-1}\}$, (if i = 1, then $S_2=\emptyset$) lying at a distance less than k to u_i, for all $1\le i\le k-1$. Then

$$C_n(v_i,P_p^c,k)=\sum_{j=1}^{2}\binom{2}{j}\binom{2k+2i-4}{n-j-1}=\binom{2k+2i-2}{n-1}-\binom{2k+2i-4}{n-1},$$

$1\le i\le k$.

Hence,

$$\sum_{i=1}^{k}C_n(v_i,P_p^c,k)=\sum_{i=1}^{k}\left[\binom{2k+2i-2}{n-1}-\binom{2k+2i-4}{n-1}\right]=\binom{4k-2}{n-1}-\binom{2k-2}{n-1}.$$

And,

$$C_n(u_i,P_p^c,k)=\sum_{j=1}^{2}\binom{2}{j}\binom{2k+2i-4}{n-j-1}=\binom{2k+2i-2}{n-1}-\binom{2k+2i-4}{n-1},$$

$1\le i\le k-1$.

Hence,

$$\sum_{i=1}^{k-1} C_n(u_i, P_p^c, k) = \sum_{i=1}^{k-1}\left[\binom{2k+2i-2}{n-1} - \binom{2k+2i-4}{n-1}\right] = \binom{4k-4}{n-1} - \binom{2k-2}{n-1}.$$

b. There are four vertices $\{u_{i-k}, v_{i-k}, u_{i+k-1}, v_{i+k}\}$ lying at a distance k from v_i, for all $k+1 \le i \le p-k$ and there are $4k-4$ vertices $\{v_{i-k+1}, \ldots, v_{i+k-1}\} \cup \{u_{i-k+1}, \ldots, u_{i+k-2}\} - \{v_i\}$ lying at a distance less than k to v_i. Also, there are four vertices $\{v_{i-k+1}, u_{i-k+1}, u_{i+k-1}, v_{i+k}\}$ lying at a distance k from u_i, for all $k \le i \le p-k$ and there are $4k-6$ vertices $\{v_{i-k+2}, \ldots, v_{i+k-1}\} \cup \{u_{i-k+2}, \ldots, u_{i+k-2}\} - \{u_i\}$ lying at a distance less than k to u_i. Then

$$C_n(v_i, P_p^c, k) = \sum_{j=1}^{4}\binom{4}{j}\binom{4k-4}{n-j-1} = \binom{4k}{n-1} - \binom{4k-4}{n-1}, \text{ for all}$$

$k+1 \le i \le p-k$.

And,

$$C_n(u_i, P_p^c, k) = \sum_{j=1}^{4}\binom{4}{j}\binom{4k-6}{n-j-1} = \binom{4k-2}{n-1} - \binom{4k-6}{n-1}, \text{ for all}$$

$k \le i \le p-k$.

Case II : For $k = \left\lfloor \frac{p}{2} \right\rfloor$ we have:

If p is an odd number, then the proof of this case as the same of Case I (a and b) and when p is an even number, then we obtain the same subcase (a) from Case I but there is only one vertex (say , $u_{\frac{p}{2}}$) satisfying the subcase (b) from Case I.

Case III : For $\left\lfloor \frac{p}{2} \right\rfloor + 1 \le k \le p-1$, There are two vertices v_{i+k}, u_{i+k-1} lying at a distance k from v_i (or u_i), for all $1 \le i \le p-k$ and there are $2k+2i-4$ vertices $S_1 \cup \{v_{i+1}, \ldots, v_{i+k-1}\} \cup \{u_1, \ldots, u_{i+k-2}\}$, where $S_1 = \{v_1, \ldots, v_{i-1}\}$, (if $i = 1$, then $S_1 = \varnothing$) lying at a distance less than k to v_i. Also, there are $2k+2i-4$ vertices $S_2 \cup \{u_{i+1}, \ldots, u_{i+k-2}\} \cup \{v_1, \ldots, v_{i+k-1}\}$, where $S_2 = \{u_1, \ldots, u_{i-1}\}$, (if $i = 1$, then $S_2 = \varnothing$) lying at a distance less than k to u_i. Then

$$C_n(v_i, P_p^c, k) = C_n(u_i, P_p^c, k) = \sum_{j=1}^{4}\binom{2}{j}\binom{2k+2i-4}{n-j-1}$$

$$= \binom{2k+2i-2}{n-1} - \binom{2k+2i-4}{n-1}, \quad 1 \le i \le p-k.$$

Hence, $\sum_{i=1}^{p-k} C_n(v_i, P_p^c, k) = \sum_{i=1}^{p-k} C_n(u_i, P_p^c, k)$

$$=\sum_{i=1}^{p-k}\left[\binom{2k+2i-2}{n-1}-\binom{2k+2i-4}{n-1}\right]=\binom{2p-2}{n-1}-\binom{2k-2}{n-1}.$$

Hence, from three cases and by the $M_n -$ polynomial of a vertex property, we have

$$C_n(P_p^c,k)=\binom{p-2k}{1}\left\{\binom{4k}{n-1}-\binom{4k-4}{n-1}\right\}+\binom{p-2k+1}{1}\left\{\binom{4k-2}{n-1}-\binom{4k-6}{n-1}\right\}$$

$$+\begin{cases} 2\binom{4k-4}{n-1}+2\binom{4k-2}{n-1}-4\binom{2k-2}{n-1}; 2\le k\le \left\lfloor \frac{p}{2}\right\rfloor, \\ 4\binom{2p-2}{n-1}-4\binom{2k-2}{n-1}; \left\lfloor \frac{p}{2}\right\rfloor+1\le k\le p-1. \end{cases}$$

We note that :

$$M_n(P_4^c;x)=\left[5\binom{2}{n-1}+2\binom{4}{n-1}\right]x+\left[3\binom{6}{n-1}+2\binom{4}{n-1}-5\binom{2}{n-1}\right]x^2$$

$$+4\left[\binom{6}{n-1}-\binom{4}{n-1}\right]x^3,$$

and

$$M_n(P_5^c;x)=\left[6\binom{2}{n-1}+3\binom{4}{n-1}\right]x+\left[\binom{8}{n-1}+4\binom{6}{n-1}+\binom{4}{n-1}-6\binom{2}{n-1}\right]x^2$$

$$+4\left[\binom{8}{n-1}-\binom{4}{n-1}\right]x^3+4\left[\binom{8}{n-1}-\binom{6}{n-1}\right]x^4$$

are satisfied the formula of the Theorem 5.1. With this the proof ends. □

Corollary 5.2 : *For all* $p\ge 4$, *we have* :

$$M_n(P_p^c)=(p+1)\binom{2}{n-1}+(p-2)\binom{4}{n-1}+\sum_{k=2}^{\left\lfloor \frac{p}{2}\right\rfloor}k\{(p-2k)\left[\binom{4k}{n-1}-\binom{4k-4}{n-1}\right]$$

$$+(p-2k+1)\left[\binom{4k-2}{n-1}-\binom{4k-6}{n-1}\right]+2\binom{4k-4}{n-1}+2\binom{4k-2}{n-1}-4\binom{2k-2}{n-1}\}$$

$$+4\sum_{k=\left\lfloor \frac{p}{2}\right\rfloor+1}^{p-1} k\left\{\binom{2p-2}{n-1}-\binom{2k-2}{n-1}\right\}.$$

□

6. Cog-Cycle Graph

Definition : Let $C_p : v_1, v_2, \ldots, v_p, v_1,\ p \geq 3,$ be a cycle of order p. The cog-cycle $C_p^c,$ is obtained from Cp by adding p new vertices $U = \{u_1, u_2, \ldots, u_p\},$ and $2p$ edges $\{u_i v_i, u_i v_{i+1} : i = 1, 2, \ldots, p\},\ v_{p+1} \equiv v_1.$

Some Properties of A Cog-Cycle Graph C_p^c :

1. **The order and the size :** $p(C_p^c) = 2p$ and $q(C_p^c) = 3p.$
2. **The max -*n*- diameter :** $\delta(C_p^c) = \left\lfloor \frac{p}{2} \right\rfloor + 1,$ for all $p \geq 6.$
3. **M_n – polynomial of a vertex :** For all $1 \leq i \leq p$ the vertices v_i have the same the M_n – polynomial of a vertex in a graph C_p^c. Also, the vertices u_i.
4. **The max – *n* – distance :** The vertices of the graph C_p^c have the max – n – distance between the vertex v and any subset S of vertices of $V(C_p^c),\ |S| = n-1,\ 3 \leq n \leq 2p,\ p \geq 4,$ as following:

$$d_{\max}(v_i, S) \leq \left\lceil \frac{p}{2} \right\rceil,\ \text{for all } 1 \leq i \leq p.$$

$$d_{\max}(u_i, S) \leq \left\lfloor \frac{p}{2} \right\rfloor + 1,\ \text{for all } 1 \leq i \leq p.$$

Theorem 6.1 : *Let C_p^c be a cog-cycle graph of order $2p, p \geq 6$. Then for all $3 \leq n \leq 2p,$ we have :*

$$M_n(C_p^c; x) = p\left[\binom{2}{n-1}+\binom{4}{n-1}\right]x$$

$$+p\sum_{k=2}^{\left\lfloor \frac{p}{2}\right\rfloor-1}\left[\binom{4k}{n-1}+\binom{4k-2}{n-1}-\binom{4k-4}{n-1}-\binom{4k-6}{n-1}\right]x^k + p\alpha x^{\left\lfloor \frac{p}{2}\right\rfloor} + +p\beta x^{\left\lfloor \frac{p}{2}\right\rfloor+1},$$

where

$$\alpha=\begin{cases}\binom{2p-1}{n-1}+\binom{2p-2}{n-1}-\binom{2p-4}{n-1}-\binom{2p-6}{n-1}, p \text{ is an even,}\\ \binom{2p-2}{n-1}+\binom{2p-4}{n-1}-\binom{2p-6}{n-1}-\binom{2p-8}{n-1}, p \text{ is an odd.}\end{cases}$$

$$\beta=\begin{cases}\binom{2p-1}{n-1}-\binom{2p-2}{n-1}, & p \text{ is an even,}\\ 2\binom{2p-1}{n-1}-\binom{2p-2}{n-1}-\binom{2p-4}{n-1}, & p \text{ is an odd.}\end{cases}$$

***Proof*:** It is obvious that $C_n(C_p^c,1)=p\left[\binom{2}{n-1}+\binom{4}{n-1}\right]$. Let $U\cup V$ be the set of vertices of C_p^c, where $V=\{v_i : i=1,2,\dots,p\}$ be the set of the vertices of the mean cycle C_p and $U=\{u_i : i=1,2,\dots,p\}$ be the set of the cog-vertices of a cycle. To prove $C_n(C_p^c,k)$ for all $2\le k\le\left\lfloor\frac{p}{2}\right\rfloor+1$, it is sufficient to prove the coefficients $C_n(u_1,C_p^c,k)$ and $C_n(v_1,C_p^c,k)$ for all $2\le k\le\left\lfloor\frac{p}{2}\right\rfloor+1$ and by using the M_n-polynomial of a vertex property we get the required result, let S be a subset of vertices of C_p^c has length $n-1, n\ge 3$. Then we have three cases:

Case I : For all $2\le k\le\left\lfloor\frac{p}{2}\right\rfloor-1$, there are four vertices $\{v_{k+1},v_{p-k+1}, u_k,u_{p-k+1}\}$ lying at a distance k from v_1 and there are $4k$ - 4 vertices $\{v_2,\dots,v_k,v_p,\dots,v_{p-k+2},u_1,\dots,u_{k-1},u_p,\dots,u_{p-k+2}\}$ lying at a distance less than k to v_1. And there are $\{v_{k+1},v_{p-k+2},u_k,u_{p-k+2}\}$ lying at a distance k from u_1 and there are $4k$ - 6 vertices $\{v_1,\dots,v_k\}\cup S_1$, $S_1=\{v_p,\dots,v_{p-k+3},u_2,\dots,u_{k-1},u_p,\dots,u_{p-k+3}\}$ lying at a distance less than k to u_1, where $S_1=\varnothing$ when $k=2$. Then

$$C_n(v_1,C_p^c,k)=\sum\nolimits_{j=1}^{4}\binom{4}{j}\binom{4k-4}{n-j-1}=\binom{4k}{n-1}-\binom{4k-4}{n-1}, \text{ and}$$

$$C_n(u_1,C_p^c,k)=\sum\nolimits_{j=1}^{4}\binom{4}{j}\binom{4k-6}{n-j-1}=\binom{4k-2}{n-1}-\binom{4k-6}{n-1}.$$

Case II : For $k=\left\lfloor\frac{p}{2}\right\rfloor$.

When p is an even number, then there are three vertices $\{v_{\frac{p}{2}+1}, u_{\frac{p}{2}}, u_{\frac{p}{2}+1}\}$ lying at a distance $k = \frac{p}{2}$ from v_1 and there are $2p-4$ vertices $\{v_2, \ldots, v_{\frac{p}{2}}, v_p, \ldots, v_{\frac{p}{2}+2}, u_1, \ldots, u_{\frac{p}{2}-1}, u_p, \ldots, u_{\frac{p}{2}+2}\}$ lying at a distance less than $\frac{p}{2}$ to v_1. And there are $\{v_{\frac{p}{2}+1}, v_{\frac{p}{2}+2}, u_{\frac{p}{2}}, u_{\frac{p}{2}+2}\}$ lying at a distance $\frac{p}{2}$ from u_1 and there are $2p-6$ vertices $\{v_1, \ldots, v_{\frac{p}{2}}, v_p, \ldots, v_{\frac{p}{2}+3}, u_2, \ldots, u_{\frac{p}{2}-1}, u_p, \ldots, u_{\frac{p}{2}+3}\}$ lying at a distance less than $\frac{p}{2}$ to u_1. Then

$$C_n\left(v_1, C_p^c, \frac{p}{2}\right) = \sum_{j=1}^{3} \binom{3}{j}\binom{2p-4}{n-j-1} = \binom{2p-1}{n-1} - \binom{2p-4}{n-1}, \text{ and}$$

$$C_n\left(u_1, C_p^c, \frac{p}{2}\right) = \sum_{j=1}^{4} \binom{4}{j}\binom{2p-6}{n-j-1} = \binom{2p-2}{n-1} - \binom{2p-6}{n-1}.$$

When p is an odd number, the proof as the same of Case I.

Case III : For $k = \left\lfloor \frac{p}{2} \right\rfloor + 1.$

When p is an even number, there is no vertex lying at a distance $k = \frac{p}{2}+1$ from v_1. But there is only one vertex $u_{\frac{p}{2}+1}$ lying at a distance $k = \frac{p}{2}+1$ from u_1, and there are $2p-2$ vertices lying at a distance less than $\frac{p}{2}+1$ to u_1.

$$C_n\left(u_1, C_p^c, \frac{p}{2}+1\right) = \sum_{j=1}^{1} \binom{1}{j}\binom{2p-2}{n-j-1} = \binom{2p-1}{n-1} - \binom{2p-2}{n-1}$$

When p is an odd number, there is only one vertex $u_{\frac{p+1}{2}}$ lying at a distance $k = \frac{p+1}{2}$ from v_1 and there are $2p-2$ vertices lying at a distance less than $\frac{p+1}{2}$ to v_1. And there are three vertices $\{v_{\frac{p+3}{2}}, u_{\frac{p+1}{2}}, u_{\frac{p+3}{2}}\}$ lying at a distance $\frac{p+1}{2}$ from u_1 and there are $2p-4$ vertices lying at a distance less than $\frac{p+1}{2}$ to u_1. Then

$$C_n\left(v_1, C_p^c, \frac{p+1}{2}\right) = \sum_{j=1}^{1} \binom{1}{j}\binom{2p-2}{n-j-1} = \binom{2p-1}{n-1} - \binom{2p-2}{n-1}, \text{ and}$$

$$C_n\left(u_1, C_p^c, \frac{p+1}{2}\right) = \sum_{j=1}^{3} \binom{3}{j}\binom{2p-4}{n-j-1} = \binom{2p-1}{n-1} - \binom{2p-4}{n-1}.$$

Since $C_n(C_p^c,k)=\sum_{i=1}^{p}C_n(v_i,C_p^c,k)+\sum_{i=1}^{p}C_n(u_i,C_p^c,k),$ and by using the M_n- polynomial of a vertex property, we get the required formula. ☐

Corollary 6.2 : *For all* $p\geq 6,$ *we have* :

$$M_n(C_p^c)=p\left[\binom{2}{n-1}+\binom{4}{n-1}\right]+p\left\lfloor\tfrac{p}{2}\right\rfloor\alpha+p\left(\left\lfloor\tfrac{p}{2}\right\rfloor+1\right)\beta$$

$$+p\sum_{k=2}^{\left\lfloor\frac{p}{2}\right\rfloor-1}k\left[\binom{4k}{n-1}+\binom{4k-2}{n-1}-\binom{4k-4}{n-1}-\binom{4k-6}{n-1}\right],$$

where

$$\alpha=\begin{cases}\binom{2p-1}{n-1}+\binom{2p-2}{n-1}-\binom{2p-4}{n-1}-\binom{2p-6}{n-1}, p \text{ is an even,}\\ \binom{2p-2}{n-1}+\binom{2p-4}{n-1}-\binom{2p-6}{n-1}-\binom{2p-8}{n-1}, p \text{ is an odd.}\end{cases}$$

$$\beta=\begin{cases}\binom{2p-1}{n-1}-\binom{2p-2}{n-1}, & p \text{ is an even,}\\ 2\binom{2p-1}{n-1}-\binom{2p-2}{n-1}-\binom{2p-4}{n-1}, & p \text{ is an odd.}\end{cases}$$

***Proof* :** Obvious. ☐

Conclusion

We can get Hosoya polynomial and wiener index for some special cog-graphs from

M_n - polynomial and M_n - index respectively and which have important applications in chemistry, engineering, computers and the rest of the sciences by substituting in the value of $n=2$ and then dividing M_2 - polynomial and M_2 - index by 2. (see Table 1).

Table 1

cog-graph	**Hosoya polynomial**	**Wiener index**
K_p^c	$\frac{p(p+3)}{2}x + p(p-1)x^2 + \frac{p(p-3)}{2}x^3$	$p(4p-5)$
S_p^c	$(p-1)\left\{3x + \frac{p+2}{2}x^2 + (p-3)x^3 + \frac{p-4}{2}x^4\right\}$	$6(p-1)(p-2)$
W_p^c	$4(p-1)x + \frac{(p-1)(p+4)}{2}x^2 + (p-1)(p-4)x^3 + \frac{(p-1)(p-6)}{2}x^4$	$2(p-1)(3p-8)$
P_p^c	$3(p-1)x + 4\sum_{k=2}^{p-1}(p-k)x^k$	$\frac{1}{3}(p-1)(2p^2+2p-3)$
C_p^c	$3p\,x + 4p\sum_{k=2}^{\left\lfloor \frac{p}{2}\right\rfloor -1} x^k + p\begin{cases} 4x^{\frac{p-1}{2}} + 2x^{\frac{p+1}{2}}, \text{if } p \text{ is odd} \\ \frac{7}{2}x^{\frac{p}{2}} + \frac{1}{2}x^{\frac{p}{2}+1}, \text{if } p \text{ is even} \end{cases}$	$\frac{1}{2}p(p^2+2p-1)$

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References

[1] P. Dankelmann, W. Goddard, M. A. Henning, H. C. Swart, Generalized Eccentricity, Radius and Diameter in Graphs; Foundation for Research Development, John Wiely & Sons, Inc. Networks 34, (1999) 312–319.

[2] G. Chartrand, L. Lesniak, Graphs and Digraphs, 6th ed. Wadsworth and Brooks / Cole, California, (2016).

[3] F. Buckley and F. Harary, Distance in Graphs. Addison -Wesley, Redwood, California, (1990).

[4] R. A. Mustafa, A. M. Ali and A.M. Khidhir, M_n-polynomials of some special graphs, *Iraqi Journal of Science,* 62:6 (2021),1986–1993.

[5] R. A. Mustafa, A. M. Ali and A.M. Khidhir, M_n- Polynomials of general thorn path graph, *Journal of Physics: Conference Series*, 1897:1(2021), 1-10

[6] A. M. Ali , A. S. Aziz, A Relation between D-Index and Wiener Index for r-Regular Graphs. *International Journal of Mathematics and Mathematical Sciences*, 2020 (2020) 1-6.

[7] A. A. Kinsley, P. S. Ananthi, Dd – Distance in Graphs , Imperial Journal of Interdisciplinary Research, 3 (2017) 1457 – 1459.

[8] A. M. Ali and A. A. Ali, The Connected Detour Numbers of Special Classes of Connected Graphs, *Journal of Mathematics*, 2019(2019), 1–9.

[9] M. V. Diudea, QSPR/QSAR Studies by Molecular Descriptors, Nova, Huntington, New York, (2001).

[10] A. M. Khidhir, A.M. Ali and S. M. Aziz,Application of width distance on semi–star link satellite constellation, *Journal of Discrete Mathematical Sciences and Cryptography*, 24:3(2021)797–807.

[11] H. Hosoya, On some counting polynomials in chemistry. Discrete Applied Math., 19(1988). 239-257.

[12] K. G. Sreekumar and K. Manilal (2018) Hosoya polynomial and Harary index of SM family of graphs, *Journal of Information and Optimization Sciences*, 39:2 (2018), 581 – 590.

[13] M. M. Abdullah and A. M. Ali, Schultz and Modified Schultz Polynomials for Edge - Identification Chain and Ring - For Pentagon and Hexagon Graphs, *Journal of Physics: Conference Series*, 1818:1 (2021), 1-20.

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